ELL 333

30.08.2019

Example: Multi-input case

$\dot{x} = Ax + B\underbrace{u}_{2\times 1}$

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

Can we design $u = -Kx$ to arbitrarily place eigenvalues? If so, what is K?

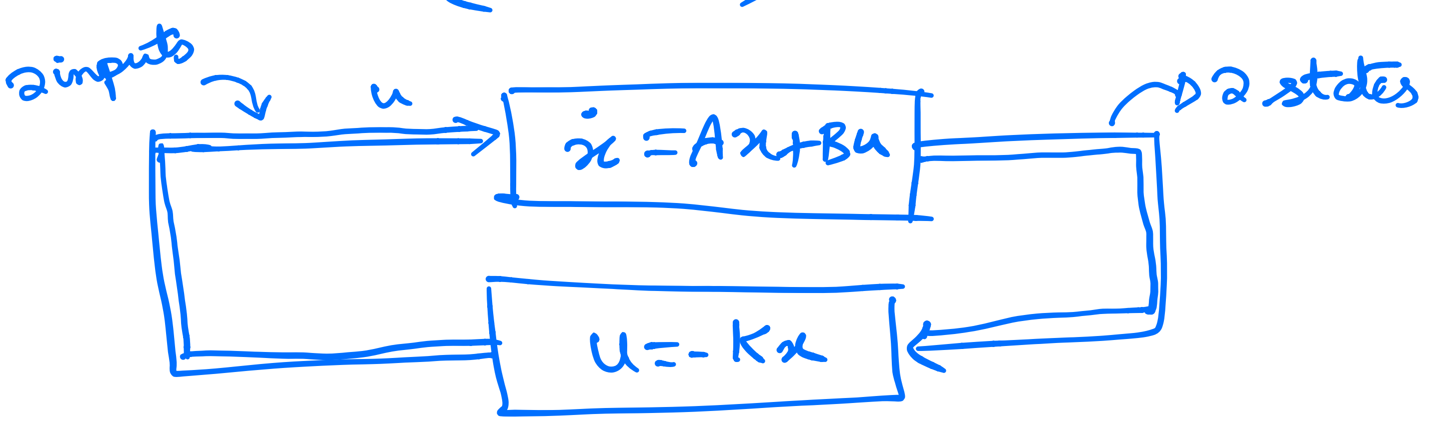
$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_{2\times 2}$$

2 eigenvalues to be placed, 4 unknowns

Direct Method

$\dot{x} = Ax + Bu, \quad u = -Kx$

$\Rightarrow \dot{x} = (A - BK)x$



$$A - BK = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} - \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -k_{11} & 1-k_{12} \\ \Omega^2 - k_{21} & -k_{22} \end{bmatrix}$$

charpoly $(A-BK)$: $(s+k_{11})(s+k_{22}) - (1-k_{12})(\Omega^2 - k_{21})$

$$s^2 + (k_{11}+k_{22}) s + (k_{11}k_{22} - (1-k_{12})(\Omega^2 - k_{21})) = 0$$

Suppose desired eigenvalues : α, β

$\Rightarrow$ we want characteristic polynomial to be

$$(s-\alpha)(s-\beta) = 0$$

$$s^2 - (\alpha+\beta) s + \alpha\beta = 0$$

Equate co-efficients

$$k_{11} + k_{22} = -(\alpha+\beta)$$

$$k_{11}k_{22} - (1-k_{12})(\Omega^2 - k_{21}) = \alpha\beta$$

This admits multiple solutions.

How to choose among multiple choices of K?

Soln 1: $k_{11} = -\alpha$, $k_{22} = -\beta$

$k_{12} = 1$ or $k_{21} = \Omega^2$

Soln 2: $k_{22} = 0$, $k_{11} = -(\alpha+\beta)$

$k_{12} = 0$, $k_{21} = \Omega^2 + \alpha\beta$

Theorem (no proof here)

(A, B) is controllable (B can be multi-input)

$\Longleftrightarrow$ eigenvalues of $A - BK$ can be arbitrarily assigned.

For single input B, we have shown

(A, B) is controllable

$\Rightarrow$ eigenvalues of $A - BK$ can be arbitrarily assigned.

Quiz 4

Show that

(A, B) is controllable

$\left(\text{rank}\{[B \;\; AB \;\; \ldots \;\; A^{n-1}B]\} = n \right)$

$$\Rightarrow \quad \text{rank}\,[\underbrace{\lambda I - A}_{n\times n} \;\vdots\; \underbrace{B}_{n\times m}] = n$$

for all complex numbers λ.

(Hint: if λ is not an eigenvalue of A, then $\text{rank}\{[\lambda I - A]\} = n$ automatically. Therefore, check condition for eigenvalue λ)