ELL333
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Controllability

Minor 1, Q1 : Bicycle

Numerical values of $A_{4\times4}$, $B_{4\times2}$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.5 & -4.1 & -0.2 & -0.7 \\ 12.3 & 21.5 & 7.7 & -6.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

MATLAB:
eig(A) $\quad 2.6 \pm 1.7j$
$\quad -3.1$
$\quad -8.6$

Goal: design feedback $u = -Kx$, to make system stable

Can it be done?
check rank $\{[B \;\; AB \;\; A^2B \;\; A^3B]\} = 4$

MATLAB: rank, ctrb(A, B)

In this case, rank = 4
$\Rightarrow$ controllable.

Controllable with one input?

$$Bu = \left[\begin{array}{c|c} B_1 & B_2 \end{array}\right]_{4\times 2} \begin{bmatrix} u_1 \\ \hline u_2 \end{bmatrix}_{2\times 1}$$

$u_1 = 0$

$$\longrightarrow \begin{bmatrix} B_2 \end{bmatrix}_{4\times 1} u_2$$

$$\text{rank}\{[B_2 \quad AB_2 \quad A^2B_2 \quad A^3B_2]\} = 4$$

Place poles as desired if controllable

MATLAB: place (single input / multiple input)
ackermann (single input)

$p = [-1, -2, -3, -4]$

$$\longrightarrow K = \begin{bmatrix} 21.5 & -4.1 & 6.8 & -0.7 \\ 12.3 & 23.5 & 7.7 & -3.3 \end{bmatrix}$$

$p = [-10, -20, -30, -40]$

$$\longrightarrow K = \begin{bmatrix} 1209.5 & -4.1 & 69.0 & -0.7 \\ 12.3 & 221.5 & 7.7 & 25.7 \end{bmatrix}$$

Show that

(A, B) is controllable

($\mathrm{rank}\{[B \ AB \ \dots \ A^{n-1}B]\} = n$)

$$\Rightarrow \quad \mathrm{rank}[\overbrace{\lambda I - A}^{n\times n} \ \vdots \ \underbrace{B}_{}^{n\times m}] = n$$

for all complex numbers λ.

(Hint: if λ is not an eigenvalue of A, then $\mathrm{rank}\{[\lambda I - A]\} = n$ automatically. Therefore, check condition for eigenvalue λ)

(A, B) controllable $\Rightarrow$ $\mathrm{rank}[\lambda I - A \ \vdots \ B] = n$

$\mathrm{rank}\{[B \ AB \dots A^{n-1}B]\} = n$ $\quad \forall \lambda \in \mathbb{C}$

We are only checking for eigenvalues λ

Suppose $\mathrm{rank}\{\underbrace{[\lambda I - A \ \vdots \ B]}_{n\times(n+m)}\} < n$

$\exists$ non-zero vector x such that ?

$$x' [\lambda I - A \ \vdots \ B] = 0$$

$$\Rightarrow [x'(\lambda I - A) \ \vdots \ x'B] = 0$$

$$\Rightarrow \quad x'(\lambda I - A) = 0 \quad \text{and} \quad \underline{x'B = 0}$$

$$x'(\lambda I - A) = 0 \Rightarrow \underline{x'A = \lambda x'}$$

$$x'B = 0, \quad x'A = \lambda x'$$

What can we conclude about rank $\{[B \;\; AB \;\dots\; A^{n-1}B]\}$?

$$x'B = 0 \qquad x'A = \lambda x'$$

$$x'AB = \lambda x'B = 0$$

$$x'A^2B = \lambda x'AB = 0$$

$$\vdots$$

$$x'A^{n-1}B = \lambda x' A^{n-2}B = 0$$

$$\Rightarrow x'[B \;\; AB \;\dots\; A^{n-1}B] = 0$$

This is a contradiction

Interesting idea

$$(A, B) \xrightarrow{T} (A_c, B_c) \quad \text{companion form}$$

$$[\lambda I - A_c \;\; B_c] = \left[\begin{array}{ccccc|c} \lambda + a_1 & a_2 & \dots & & a_n & 1 \\ -1 & \lambda & \dots & & 0 & 0 \\ 0 & -1 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & & \lambda & 0 \end{array}\right]$$

$$\operatorname{rank}\left\{\left[B \;\; AB \;\; \cdots \;\; A^{n-1}B\right]\right\} = r < n$$

$\Longleftrightarrow$ there exists an invertible transformation T, such that

$$T A T^{-1} = \tilde{A} = \begin{array}{cc} \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline 0 & \tilde{A}_{22} \end{array}\right] & \begin{array}{l} \updownarrow r \\ \updownarrow n-r \end{array} \\ \begin{array}{cc} \longleftrightarrow & \longleftrightarrow \\ r & n-r \end{array} & \end{array}$$

$$TB = \tilde{B} = \begin{array}{cc} \left[\begin{array}{c} \tilde{B}_1 \\ \hline 0 \end{array}\right] & \begin{array}{l} \updownarrow r \\ \updownarrow n-r \end{array} \end{array}$$

and $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable.

i.e $\operatorname{rank}\left\{\left[\tilde{B}_1 \;\; \tilde{A}_{11}\tilde{B}_1 \;\; \tilde{A}_{11}^2\tilde{B}_1 \;\; \cdots \;\; \tilde{A}_{11}^{r-1}\tilde{B}_1\right]\right\} = r$