ELL 333
06.09.2019

How to get state (x) from output (y)?
(so that in can be used for $u=-Kx$)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

ignoring why?

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$$

$$\Rightarrow y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau + Du(t)$$

output we can measure

we know system parameter

don't know

input we give

we know system parameter

We ignore B & D terms when we want to estimate x from y, because?

$$\underbrace{y(t) - C\int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau - Du(t)}_{\text{known effectively,}} = Ce^{At}x_0$$

Only unknown

so we take input $u=0$ (or $B=D=0$) so that those terms not there

if we get x_0 then $x(t) = e^{At}x_0$ when $u=0$

$\therefore$ We simplify to the question of determining x_0 from y using the equation

$$y = C e^{At} x_0$$

Observability

The system $\dot{x} = Ax$, $y = Cx$ is observable if it is possible to find any initial state x_0 $(x(t=0))$ from observations of output $y(t)$ $0 \le t \le T$ over finite duration $[0, T]$.

Nullspace or Kernel

Suppose $U: \mathbb{R}^m \to \mathbb{R}^n$, then

$$Ker(U) = \{ x \in \mathbb{R}^m \text{ such that } Ux = 0 \}$$

$$y = C e^{At} x_0$$

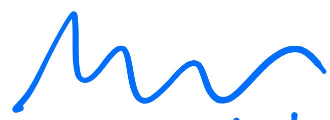
Can we reduce observing x_0 from y in terms of $Ker\{C e^{At}\}$?

$$\underset{(\text{state})}{C e^{At} : \mathbb{R}^n} \longrightarrow \underset{(\text{output})}{\mathbb{R}^p}$$

Suppose there was $v\ (\neq 0)$ in $\operatorname{Ker}(Ce^{At})$, then can we get x_0 from y?

$$v \in \operatorname{Ker}(Ce^{At}) \Rightarrow Ce^{At}v = 0$$

must have been some x_0 here



$y(t) = Ce^{At}x_0$

If $v\ (\neq 0) \in \operatorname{Ker}(Ce^{At})$,

$$Ce^{At}(x_0 + v) = Ce^{At}x_0 + Ce^{At}v$$
$$= y + 0$$

Why I am discussing this before going to theorems showing observability $\Longleftrightarrow$ $\operatorname{rank}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = n$

$$u \xrightarrow{\text{convolution}} x \xrightarrow{Ce^{At}} y$$

Theorem: $\operatorname{Ker}(Ce^{At}) = \operatorname{Ker}(C) \cap \operatorname{Ker}(CA) \cap \operatorname{Ker}(CA^2) \cap \cdots \operatorname{Ker}(CA^{n-1})$

Proof:

$$\mathrm{Ker}(Ce^{At}) \subseteq \mathrm{Ker}(C) \cap \mathrm{Ker}(CA) \dots \cap \mathrm{Ker}(CA^{n-1})$$

Suppose $v \in \mathrm{Ker}\{Ce^{At}\}$ $\quad$ $\underline{t=0}$

$\Rightarrow Ce^{At}v = 0 \rightarrow Cv = 0$

$\Rightarrow v \in \mathrm{Ker}(C)$

Differentiate w.r.t t

$\Rightarrow CAe^{At}v = 0 \rightarrow CAv = 0$

$\Rightarrow v \in \mathrm{Ker}(CA)$

Differentiate w.r.t t

$\Rightarrow CA^2e^{At}v = 0 \rightarrow CA^2v = 0$

$\Rightarrow v \in \mathrm{Ker}(CA^2)$

$\therefore v \in \mathrm{Ker}(C) \cap \mathrm{Ker}(CA) \dots \cap \mathrm{Ker}(CA^{n-1})$

$$\mathrm{Ker}(C) \cap \mathrm{Ker}(CA) \dots \mathrm{Ker}(CA^{n-1}) \subseteq \mathrm{Ker}(Ce^{At})$$

Suppose $x \in \mathrm{Ker}(C) \cap \mathrm{Ker}(CA) \dots \mathrm{Ker}(CA^{n-1})$

$\Rightarrow Cx = 0,\ CAx = 0, \dots CA^{n-1}x = 0$

How to show $Ce^{At}x = 0$?

$$e^{At} = f_0(t)I + f_1(t)A + \frac{f_2(t)}{2!}A^2 + \dots + \frac{f_{n-1}(t)}{(n-1)!}A^{n-1}$$

(because of Cayley-Hamilton Thm and matrix exponential

$$\therefore\ Ce^{At} = f_0(t).C + f_1(t).CA + \dots + \frac{f_{n-1}(t)}{(n-1)!} CA^{n-1}$$

$$\Rightarrow Ce^{At}x = 0$$

$$\Rightarrow x \in \operatorname{Ker}(Ce^{At})$$

$$\therefore \operatorname{Ker}(Ce^{At}) = \operatorname{Ker}(C) \cap \operatorname{Ker}(CA) \dots \operatorname{Ker}(CA^{n-1})$$

$$\operatorname{Ker}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np \times n}$$

$$C : \mathbb{R}^n \to \mathbb{R}^p$$

$$CA : \mathbb{R}^n \to \mathbb{R}^p$$

$$CA^{n-1} : \mathbb{R}^n \to \mathbb{R}^p$$

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np \times n} x_{n \times 1} = \begin{bmatrix} Cx \\ CAx \\ \vdots \\ CA^{n-1}x \end{bmatrix}_{np \times 1}$$