ELL 333

07.09.2019

$$\boxed{y = C e^{At} x_0}$$

When can we get x_0 from y?

MAP

Observability $\xleftrightarrow{?}$ $\text{rank}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = n$

rank. nullity theorem

$\ker\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = 0$

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$W(t) : \int_0^t e^{A'z} C' C e^{Az} dz$

(Grammian)

W is ~~singular~~ non-singular $\xleftrightarrow{1}$ $\ker\{C e^{At}\} = 0$

Where does W come from?

$$y = C e^{At} x_0$$

Multiply by $e^{A't} C'$

$$\Rightarrow e^{A't} C' y = \underbrace{e^{A't} C' C e^{At}}_{\text{When is this invertible?}} x_0$$

Integrate $\int_0^t$

$$\Rightarrow \int_0^t e^{A'z} C' y(z) dz = \int_0^t e^{A'z} C' C e^{Az} x_0 dz$$

$$\Rightarrow \int_0^t e^{A'\tau} c' y(\tau)\, d\tau = \Big(\underbrace{\int_0^t \underset{n\times n}{e^{A'\tau}}\ \underset{n\times p}{c'}\ \underset{p\times n}{c}\ \underset{n\times n}{e^{A\tau}}\, d\tau}_{W_o}\Big) x_o$$

So if W_o was non-singular

$$x_o = W_o^{-1} \int_0^t e^{A'\tau} c' y(\tau)\, d\tau$$

Theorem:

W_o is non-singular $\Leftrightarrow$ $\mathrm{Ker}\{ce^{At}\} = 0$

$$\left(W_o = \int_0^t e^{A'\tau} c' c\, e^{A\tau}\, d\tau\right)$$

($\Rightarrow$) Given non-singular W_o

Suppose $\mathrm{Ker}\{ce^{At}\} \neq 0$

Then $\exists$ $v\ (\neq 0)$ such that $ce^{At} v = 0$

Consider $W_o v$

$$= \int_0^t e^{A'\tau} c' c\, e^{A\tau}\, d\tau \cdot v$$

$= 0 \Rightarrow W_o$ is singular, contradiction.

($\Leftarrow$) Given $\operatorname{Ker}\{Ce^{At}\} = 0$

Suppose W is singular

$\Rightarrow \exists\, v \ (\neq 0)$ such that $Wv = 0$

$\Rightarrow v' W v = 0$

$\Rightarrow v' \int_0^t e^{A'\tau} c' c e^{A\tau} d\tau \, v = 0$

$\Rightarrow \int_0^t v' e^{A'\tau} c' \, \underbrace{c e^{A\tau} v}\, d\tau = 0$

transpose

Denote $z = Ce^{At} v$

$\Rightarrow \int_0^t z' z \, d\tau = 0$

$\Rightarrow z = 0 \Rightarrow Ce^{At} v = 0$

a contradiction.

Quiz 5

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

a) Find Ce^{At} & $\mathrm{Ker}\left\{ \overset{\nearrow 2}{Ce^{At}} \right\}$

b) Find $\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}_{8\times 4}$ & $\mathrm{rank}\left\{ \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}}_{\hookrightarrow 2} \right\}$