ELL333

12.09.2019

[$\dot{x} = Ax$, $y = Cx$: Observability]

$\text{rank}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix}\right\} = n$

Theorem

(A, C) is observable $\Leftrightarrow$ W_o is non-singular

↑ same as

↳ Observability Grammian

$$W_o = \int_0^T e^{A't} C' C e^{At} dt$$

($\Leftarrow$) W_o is non-singular

from last lecture,
$$x_0 = W_o^{-1} \int_0^T e^{A't} C' y(t) dt$$

$\therefore$ If W_o was non-singular, we can get any x_0 from output measurements.

($\Rightarrow$) (A, C) is observable. Is W_o non-singular?

Let us assume W_o is singular.

$\Rightarrow \exists\ v (\neq 0)$ such that

$$v' W_o v = 0$$

$$\Rightarrow v' \int_0^T e^{A't} C' C e^{At} dt\, v = 0$$

can go directly using the result from last lecture

$$\Rightarrow \int_0^T \underbrace{v' e^{A't} c'}_{z'} \underbrace{c e^{At} v}_{z} \, dt = 0$$

$$\Rightarrow \quad z = 0$$

$$\Rightarrow \quad c e^{At} v = 0$$

that W_o is non-singular $\Leftrightarrow \ker\{c e^{At}\} = 0$

Why is this a problem for observability?

$$y = c e^{At} x_0$$

Problem is that $x_0 \xrightarrow{c e^{At}} y$

then $x_0 + v \xrightarrow{c e^{At}}$

also $x_0 + 2v \xrightarrow{c e^{At}}$

We cannot determine the initial condition that gives y.

$\therefore$ (A, C) is observable $\Leftrightarrow$ rank$\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = n$

definition $\longleftrightarrow$ algebraic test

TALL [ ]

Illy, we had done

(A, B) is controllable $\Leftrightarrow$ rank$\{[B \ AB \ldots A^{n-1}B]\} = n$

WIDE [ ]

Are they related

$$\underset{\text{(WIDE)}}{[B \ AB \ \dots \ A^{n-1}B]'} = \underset{\text{(TALL)}}{\begin{bmatrix} B' \\ (AB)' \\ \vdots \\ (A^{n-1}B)' \end{bmatrix}}$$

$$= \begin{bmatrix} B' \\ B'A' \\ \vdots \\ B'(A')^{n-1} \end{bmatrix} \qquad (A^2B)' = B'(A')^2$$

$\overline{(A, B)}$ is controllable for system $\dot{x} = Ax + Bu$

$\iff \text{rank}\{[B \ AB \ \dots \ A^{n-1}B]\} = n$

$\iff \text{rank}\left\{\begin{bmatrix} B' \\ B'A' \\ \vdots \\ B'(A')^{n-1} \end{bmatrix}\right\} = n$

$\iff (A', B')$ is observable for system $\dot{x} = A'x$, $y = B'x$

↑ Proof

Thm

$\therefore$ (A, B) is controllable $\iff$ (A', B') is observable

[What does this mean?]

Suppose system is observable,
how to observe the system?
(or) how to get state from output?

$\hookrightarrow$ we want state estimate $\hat{x}$,
because we want control law $u = -Kx$
and don't have x; we only have y.

What do we need?

- Possible ways

$$x = C^{-1} y \quad : C \text{ is not square}$$

$$x = (C'C)^{-1} C' C y$$

Is $C'C$ invertible if rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}_{1\times 4}$$

$$C' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4\times 1}, \quad C'C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank} = 1$$

$\hookrightarrow$ Trying to find A such that (A, C) is observable

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}, \quad \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 3 \end{bmatrix} \times \quad \text{rank}=2$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & {}^{3}2 & {}^{4}2 & {}^{5}2 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Want to find example where $C'C$ is not invertible but (A, C) is observable.

$C = [1 \;\; 0]$, $C' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow C'C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, which is not invertible

Suppose

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (if it was $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then x_2 is not connected to output, but in what is supposed it is)

$\Rightarrow \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ which has rank 2.

So, how to observe x from y?