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ELL 333

$$\frac{x(t+h)-x(t)}{h} = Ax(t)+Bu(t)+v(t)$$

blue A ≠ red A

Discrete-Time Kalman Filter

$\dot{x} = Ax + Bu + v$

$y = Cx + w$

$t \in \mathbb{R}$

$k \in \mathbb{Z}$

System

$$x_{k+1} = Ax_k + Bu_k + v_k$$
$$y_k = Cx_k + w_k$$

v_k, w_k are noise terms

$E\{v_k\} = 0$ $\quad E\{v_k w_k\} = 0$

$E\{w_k\} = 0$

$$E\{v_k v_j'\} = \begin{cases} 0, & k \neq j \\ R_v, & k = j \end{cases}$$

$$E\{w_k w_j'\} = \begin{cases} 0, & k \neq j \\ R_w, & k = j \end{cases}$$

Observer

this can be time-varying

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L_k(y_k - C\hat{x}_k)$$

Error $e_k = x_k - \hat{x}_k$

Error dynamics

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1}$$

$$= Ax_k + Bu_k + v_k - A\hat{x}_k - Bu_k - L_k(Cx_k + w_k - C\hat{x}_k)$$

$$= A(x_k - \hat{x}_k) + v_k - L_k w_k - L_k C(x_k - \hat{x}_k)$$

$$e_{k+1} = (A - L_k C)e_k + v_k - L_k w_k$$

$$\Rightarrow E\{e_{k+1}\} = (A - L_k C)E\{e_k\}$$

Equation for propagation of the mean.

What are our objectives with regard to error?

- error $\longrightarrow 0$ (same as before)

- additionally, we want error covariance $E\{e_k e_k'\}$ to be as small as possible.

so, we choose L_k to minimize $\{\ \}$ at each timestep.

P_k

$$P_k = E\{e_k e_k'\}$$
$$= E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)'\}$$

How does this propagate? Equation for propagation of variance.

$$P_{k+1} = E\{e_{k+1}\, e_{k+1}'\}$$

$$= E\{((A-L_kC)e_k + v_k - L_k w_k)\,((A-L_kC)e_k + v_k - L_k w_k)'\}$$

$$= E\{((A-L_kC)e_k + v_k - L_k w_k)\,(e_k'(A-L_kC)' + v_k' - w_k' L_k')\}$$

$$= E\{ (A-L_kC)e_k e_k'(A-L_kC)'$$

$$+ v_k v_k'$$

$$+ L_k w_k w_k' L_k'$$

$$+ (\ )\, e_k v_k' + (\ )\, v_k e_k'$$

$$+ (\ )\, e_k w_k' + (\ )\, w_k e_k'$$

$$+ (\ )\, \underbrace{\cancel{v_k w_k'}}_{=0} + (\ )\, \underbrace{\cancel{v_k' w_k}}_{=0} \}$$

$E\{e_k v_k'\} = 0$

$\hookrightarrow v_{k-1}, v_{k-2} \ldots v_0$

$+\ e_0$ assumed independent of v_k

$$= (A-L_kC)\, \overbrace{E\{e_k e_k'\}}^{P_k}\, (A-L_kC)'$$

$$+ \underbrace{E\{v_k v_k'\}}_{R_v} + L_k \underbrace{E\{w_k w_k'\}}_{R_w} L_k'$$

$$= (A-L_kC)\, P_k\, (A-L_kC)' + R_v + L_k R_w L_k'$$

$\therefore P_{k+1} = (A - L_k C) P_k (A - L_k C)' + R_v + L_k R_w L_k'$

To minimize RHS by choosing L_k

Scalar version

$$f(l) = (a - lc)^2 P + R_v + l^2 R_w$$

What l minimizes f?

$$= Pa^2 - 2acPl + Pc^2 l^2 + R_v + l^2 R_w$$

$$= l^2 (R_w + Pc^2) - 2acPl + R_v + Pa^2$$

$$= (R_w + Pc^2)\left(l^2 - \frac{2ac P}{(R_w + Pc^2)} l\right) + R_v + Pa^2$$

$$= (R_w + Pc^2)\left(l^2 - 2 \cdot \frac{acP}{(R_w + Pc^2)} l + \left(\frac{acP}{(R_w + Pc^2)}\right)^2 - \left(\frac{acP}{(R_w + Pc^2)}\right)^2\right)$$

$$+ R_v + Pa^2$$

$$= (R_w + Pc^2)\left(l - \frac{acP}{R_w + Pc^2}\right)^2$$

$$+ R_v + Pa^2 - \frac{a^2 c^2 P^2}{R_w + Pc^2}$$

$\therefore l = \frac{acP}{R_w + Pc^2}$ minimizes $f(l)$