ELL333, 18.10.2019

Q. Minimize P_k (a matrix) means minimize all entries?

effectively $\frac{n(n+1)}{2}$ as P_k is symmetric

In our case, having small values of all entries in P_k is good as that is an error covariance matrix. Small entries means more confidence in estimate.

Calculations in class were aimed at addressing whether the minimization really happens via 'completing the squares'.

Are we actually minimizing all entries?

'Completing the square' not completely obvious for matrices...

For 2x2 case does the answer match with brute force answer?

$$L_k = A P_k C' R_e^{-1}$$

$$\begin{bmatrix} E\{e_{1k}^2\} & E\{e_{1k}e_{2k}\} \\ E\{e_{1k}e_{2k}\} & E\{e_{2k}^2\} \end{bmatrix}$$

$$= A P_k C' (R_w + C P_k C')^{-1}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E\{e_{1k}^2\} & E\{e_{1k}e_{2k}\} \\ E\{e_{1k}e_{2k}\} & E\{e_{2k}^2\} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times$$

$$\left(R_w + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} E\{e_{1k}^2\} & E\{e_{1k}e_{2k}\} \\ E\{e_{1k}e_{2k}\} & E\{e_{2k}^2\} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^{-1}$$

$$= \frac{1}{R_w + E\{e_{1k}^2\}} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E\{e_{1k}^2\} \\ E\{e_{1k}e_{2k}\} \end{bmatrix}$$

L_k

$$\therefore \boxed{\begin{bmatrix} l_{1k} \\ l_{2k} \end{bmatrix}} = \frac{1}{R_w + E\{e_{1k}^2\}} \begin{bmatrix} a_{11} E\{e_{1k}^2\} + a_{12} E\{e_{1k}e_{2k}\} \\ a_{21} E\{e_{1k}^2\} + a_{22} E\{e_{1k}e_{2k}\} \end{bmatrix}$$

This is what is obtained on 'completing the square' in matrix form.

Brute force : Minimize these

$$E\{e_{1,k+1}^2\} = \boxed{(1,1)} + \boxed{v_{11}} + \boxed{R_w l_{1k}^2}$$

$$E\{e_{1(k+1)}e_{2(k+1)}\} = (1,2) + v_{12} + R_w l_{1k} l_{2k}$$

$$E\{e_{2(k+1)}^2\} = (2,2) + v_{22} + R_w l_{2k}^2$$

as functions of l_{1k}, l_{2k}.

$$\frac{\partial (1,1)}{\partial^2 l_{1k}}$$

$$\frac{\partial E\{e_{1,k+1}^2\}}{\partial l_{1k}} = -2(a_{11} - l_{1k}) E\{e_{1k}^2\} - 2 a_{12} E\{e_{1k}e_{2k}\} + \boxed{2 R_w l_{1k}}$$

Set this to zero $\Rightarrow$ $l_{1k} = \dfrac{a_{11} E\{e_{1k}^2\} + a_{12} E\{e_{1k}e_{2k}\}}{E\{e_{1k}^2\} + R_w}$

(magically) same as above formula!

Can check that $\dfrac{\partial^2 E\{e^2_{1,k+1}\}}{\partial l^2_{1k}} > 0$

Note that minimizing this term depends only on l_{1k}.

$$\frac{\partial E\{e^2_{2,k+1}\}}{\partial l_{2k}} = -2(a_{21} - l_{2k}) E\{e^2_{1k}\} - 2a_{22} E\{e_{1k} e_{2k}\} + 2R_w l_{2k}$$

Set this to zero $\Rightarrow$ $l_{2k} = \dfrac{a_{21} E\{e^2_{1k}\} + a_{22} E\{e_{1k} e_{2k}\}}{E\{e^2_{1k}\} + R_w}$

same as in formula!

Hopefully $\dfrac{\partial E\{e_{1k} e_{2k}\}}{\partial l_{1k}} = 0$ and $\dfrac{\partial E\{e_{1k} e_{2k}\}}{\partial l_{2k}} = 0$

give consistent results.

$$\frac{\partial E\{e_{1k} e_{2k}\}}{\partial l_{1k}} = \boxed{R_w l_{2k}} - (a_{21} - l_{2k}) E\{e^2_{1k}\} - a_{22} E\{e_{1k} e_{2k}\}$$

(same result)

$$\frac{\partial E\{e_{1k} e_{2k}\}}{\partial l_{2k}} = R_w l_{1k} - (a_{11} - l_{1k}) E\{e^2_{1k}\} - a_{12} E\{e_{1k} e_{2k}\}$$

(also same answer).

So conclusion is that brute force minimiz-
-ation also gives same answer as
'completing the square' as seen is the
2-state case.

This just checks that we are minimizing
all entries of co-variance matrix.

Q. What does it mean

$$\boxed{S :=} E\{(L_k - AP_k C' R_\epsilon^{-1}) R_\epsilon (L_k - AP_k C' R_\epsilon^{-1})'\} \geq 0 \quad ?$$

$$R_\epsilon = R_w + C P_k C'$$

P ($n\times n$) > 0 (matrix relation) means $x^T P x > 0 \quad \forall x \neq 0$ (x^T: $1\times n$, P: $n\times n$, x: $n\times 1$; scalar relation)

(positive definite)

positive semi-definite $P \geq 0$

negative semi-definite $P \leq 0$

negative definite $P < 0$

for $n = 1$, 2 $(l_k - (\,)\,)$

How should S look like for $n=2$ case?

For above $n=2$ example,

$$R_\epsilon = R_w + E\{e_{1k}^2\}$$

$$L_k - \underbrace{A P_k C' R_\epsilon^{-1}}_{\begin{bmatrix} m_{1k} \\ m_{2k} \end{bmatrix}}$$

$$= \begin{bmatrix} l_{1k} - m_{1k} \\ l_{2k} - m_{2k} \end{bmatrix}$$

, m_{1k}, m_{2k} can be obtained from above pages.

$$S = R_\epsilon \begin{bmatrix} l_{1k} - m_{1k} \\ l_{2k} - m_{2k} \end{bmatrix} \begin{bmatrix} l_{1k} - m_{1k} & l_{2k} - m_{2k} \end{bmatrix}$$

$$= R_\epsilon \begin{bmatrix} (l_{1k} - m_{1k})^2 & (l_{1k} - m_{1k})(l_{2k} - m_{2k}) \\ (l_{1k} - m_{1k})(l_{2k} - m_{2k}) & (l_{2k} - m_{2k})^2 \end{bmatrix}$$

$$x' S x = R_\ell \, x' \left[\quad \right] x$$

$$= R_\ell \, [x_1 \quad x_2] \begin{bmatrix} (l_{1k}-m_{1k})^2 x_1 + (l_{1k}-m_{1k})(l_{2k}-m_{2k}) x_2 \\ (l_{1k}-m_{1k})(l_{2k}-m_{2k}) x_1 + (l_{2k}-m_{2k})^2 x_2 \end{bmatrix}$$

$$= R_\ell \left[(l_{1k}-m_{1k})^2 x_1^2 + 2(l_{1k}-m_{1k})(l_{2k}-m_{2k}) x_1 x_2 + (l_{2k}-m_{2k})^2 x_2^2 \right]$$

$$= R_\ell \left[(l_{1k}-m_{1k}) x_1 + (l_{2k}-m_{2k}) x_2 \right]^2$$

≥ 0 for non-zero x_1, x_2

equal to zero when $l_{1k} = m_{1k}$, $l_{2k} = m_{2k}$

Quiz 10

Show that

$$(A-LC)P_k(A-LC)' + R_v + LR_wL'$$

$$= AP_kA' + R_v - AP_kC'R_\epsilon^{-1}CP_kA'$$
$$+ (L - AP_kC'R_\epsilon^{-1})R_\epsilon(L - AP_kC'R_\epsilon^{-1})',$$

where $R_\epsilon = R_w + CP_kC'$