ELL333

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Kalman Gain $\xrightarrow{\text{maybe later}}$ more relation to observability

previously: starting from observer equation and minimizing covariance at each time step.

now: starting from probabilistic consideration

Probability starting point

$\mathcal{C}$: we can have this also

$$y = x + n$$

output, that is actually measured
random variable

state we want to actually measure
may also be a random variable depending on the evolution equation of x

measurement noise
clearly a random variable

Estimation is effectively obtaining conditional probability of x given y.

$x|y$

What is this?

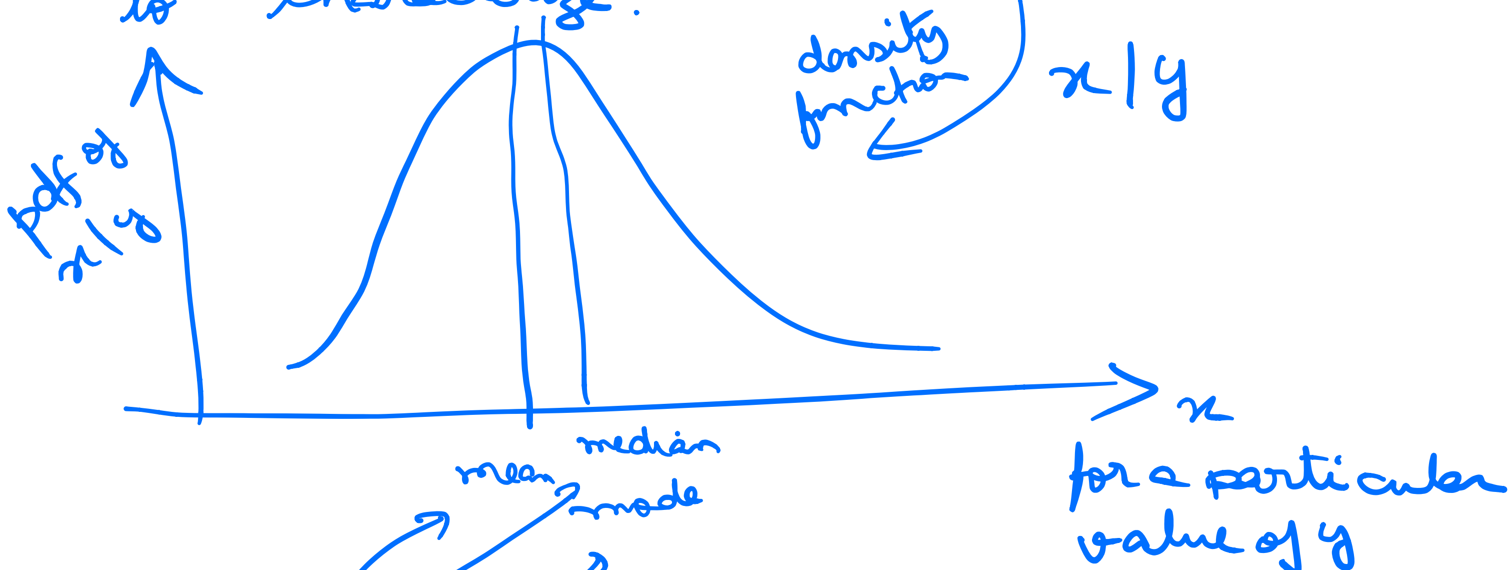
→ Random Variable

↳ requires a distribution function to characterize.

$\underbrace{\text{distribution function}}$

density function

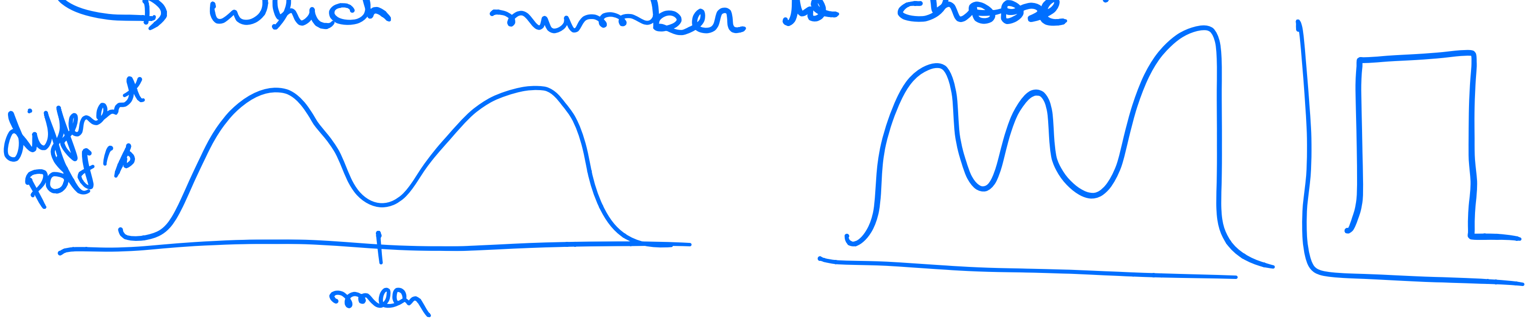
$x|y$



different numbers that summarize some aspect of pdf

↳ Does it make sense to have one number? → convenience

↳ Which number to choose?



- for "funny" pdf's mean may not be representative
  ⇒ choose mode / global maxima's
- other possibilities also exist

One possibility to find a number ($\hat{x}$)

(1) consider error square $(x - \hat{x})^2$

- as error is random variable, we take expected value of this conditioned on output y. $\underline{\underline{E}}\{(x-\hat{x})^2 \mid y\}$

- just like least square we want to take $\hat{x}$ that gives minimum value of this quantity $\min_{\hat{x}} E\{\underbrace{(x-\hat{x})^2}_{\text{vector version?}} \mid y\}$

This is called many things

- MVE : minimum variance estimate.
- MMSE : minimum mean square estimate.

What might be a guessed for $\hat{x}$?

$$E\{(x-\hat{x})^2 \mid y\} =$$

$$\int (x-\hat{x})^2 \, p_{X|Y}(x) \, dx$$

idea is to write this as

$$= (\quad) \pm (\quad) + (\hat{x} - \boxed{})^2$$

no $\hat{x}$

$$\int (x-\hat{x})^2 \, p_{X|Y}(x) \, dx$$

$$= \int (x^2 - 2x\hat{x} + \hat{x}^2) \, p_{X|Y}(x) \, dx$$

$$= \int x^2 p_{X|Y}(x)\, dx - \int 2x\hat{x} \, p_{X|Y}(x)\, dx + \int \hat{x}^2 p_{X|Y}(x)\, dx$$

$$= \int x^2 p_{X|Y}(x)\, dx - 2\hat{x} \int x \, p_{X|Y}(x)\, dx + \hat{x}^2 \underbrace{\int p_{X|Y}(x)\, dx}_{\to 1}$$

$$= \hat{x}^2 - 2\hat{x} \int x \, p_{X|Y}(x)\, dx + \int x^2 p_{X|Y}(x)\, dx + \left[\int x \, p_{X|Y}(x)\, dx\right]^2 - \left[\int x \, p_{X|Y}(x)\, dx\right]^2$$

$$= \left[\hat{x} - \int x \, p_{X|Y}(x)\, dx\right]^2 + \int x^2 p_{X|Y}(x)\, dx - \left[\int x \, p_{X|Y}(x)\, dx\right]^2$$

$$\geq \int x^2 p_{X|Y}(x)\,dx - \left[\int x\, p_{X|Y}(x)\,dx\right]^2$$

$$= E(x^2|y) - [E(x|y)]^2$$

and it takes the minimum when

$$\hat{x} = \int x\, p_{X|Y}(x)\,dx$$

$$= E(x|y)$$

conditional mean.

Proof: for scalar case

Theorem: minimum variance estimate (MVE/MMSE/...) = conditional mean