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ELL333

X is gaussian random variable

$\int_{-\infty}^{\infty} p_X(x)\,dx = 1$

pdf, $p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\bar{x})^2}{2\sigma^2}\right)$

→ x is scalar.

X and Y are jointly gaussian

pdf, $p_{XY}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \det(\Sigma)}} \exp\left(-\frac{1}{2}\begin{bmatrix} x-\bar{x} \\ y-\bar{y}\end{bmatrix}' \Sigma^{-1} \begin{bmatrix} x-\bar{x} \\ y-\bar{y}\end{bmatrix}\right)$

(x and y are scalars)

$\Sigma = \begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_y \end{bmatrix}$, $\Sigma_{xy} = \Sigma_{yx}$

Goal: $p_{X|Y}(x) = \frac{p_{XY}(x,y)}{p_Y(y)}$ → joint density → marginal density

How to calculate this?

$P(A \cap B) = P(A|B) \cdot P(B)$

$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A \cap B)}{\sum_{\text{event}} P(B \cap \text{event})}$

$= \frac{p_{XY}(x,y)}{\int_{-\infty}^{\infty} p_{XY}(x,y)\,dx}$

$$\frac{\cancel{\frac{1}{\sqrt{(2\pi)^2(\det\Sigma)}}}\exp\left\{-\frac{1}{2}\begin{bmatrix}x-\bar{x}\\ y-\bar{y}\end{bmatrix}'\Sigma^{-1}\begin{bmatrix}x-\bar{x}\\ y-\bar{y}\end{bmatrix}\right\}}{\int_{-\infty}^{\infty}\cancel{\frac{1}{\sqrt{(2\pi)^2(\det\Sigma)}}}\exp\left\{-\frac{1}{2}\begin{bmatrix}x-\bar{x}\\ y-\bar{y}\end{bmatrix}'\Sigma^{-1}\begin{bmatrix}x-\bar{x}\\ y-\bar{y}\end{bmatrix}\right\}dy}$$

$$\begin{bmatrix}x-\bar{x}\\ y-\bar{y}\end{bmatrix}'\frac{\begin{bmatrix}\Sigma_y & -\Sigma_{xy}\\ -\Sigma_{xy} & \Sigma_x\end{bmatrix}}{\Sigma_x\Sigma_y-\Sigma_{xy}^2}\begin{bmatrix}x-\bar{x}\\ y-\bar{y}\end{bmatrix}$$

$$=\frac{\begin{bmatrix}x-\bar{x} & y-\bar{y}\end{bmatrix}}{\Sigma_x\Sigma_y-\Sigma_{xy}^2}\begin{bmatrix}\Sigma_y(x-\bar{x})-\Sigma_{xy}(y-\bar{y})\\ -\Sigma_{xy}(x-\bar{x})+\Sigma_x(y-\bar{y})\end{bmatrix}$$

$$=\frac{1}{\Sigma_x\Sigma_y-\Sigma_{xy}^2}\left(\Sigma_y(x-\bar{x})^2-2\Sigma_{xy}(x-\bar{x})(y-\bar{y})+\underbrace{\Sigma_x(y-\bar{y})^2}_{\text{forget this, it cancels with numerator}}\right)$$

$$=\frac{\Sigma_y}{\Sigma_x\Sigma_y-\Sigma_{xy}^2}\left((x-\bar{x})^2-2(x-\bar{x})\cdot\frac{\Sigma_{xy}}{\Sigma_y}(y-\bar{y})\pm\left(\frac{\Sigma_{xy}}{\Sigma_y}(y-\bar{y})\right)^2\right)$$

$$= \frac{\Sigma_y}{\Sigma_x\Sigma_y - \Sigma_{xy}^2}\left(x - \bar{x} - \frac{\Sigma_{xy}}{\Sigma_y}(y-\bar{y})\right)^2 \underbrace{- \frac{\Sigma_{xy}^2}{\Sigma_y(\Sigma_x\Sigma_y - \Sigma_{xy}^2)} \cdot (y-\bar{y})^2}_{\text{forget as it cancels}}$$

expression =

$$\frac{\exp\left\{-\frac{1}{2}\frac{\Sigma_y}{\Sigma_x\Sigma_y - \Sigma_{xy}^2}\left(x-\bar{x}-\frac{\Sigma_{xy}}{\Sigma_y}(y-\bar{y})\right)^2\right\}}{\underbrace{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\frac{\Sigma_y}{\Sigma_x\Sigma_y-\Sigma_{xy}^2}\left(x-\bar{x}-\frac{\Sigma_{xy}}{\Sigma_y}(y-\bar{y})\right)^2\right\}dx}_{\text{denominator} = \sqrt{2\pi\left(\frac{\Sigma_x\Sigma_y-\Sigma_{xy}^2}{\Sigma_y}\right)}}}$$

$\frac{1}{\sigma^2}$ σ mean

$$= \frac{1}{\sqrt{2\pi\left(\frac{\Sigma_x\Sigma_y-\Sigma_{xy}^2}{\Sigma_y}\right)}} \exp\left\{-\frac{1}{2}\frac{\left(x-\bar{x}-\frac{\Sigma_{xy}}{\Sigma_y}(y-\bar{y})\right)^2}{\frac{\Sigma_x\Sigma_y-\Sigma_{xy}^2}{\Sigma_y}}\right\}$$

This is $P_{X|Y}(x)$, conditional density

$\therefore X|Y$ is also gaussian

$$E(X|Y) = \bar{x} + \frac{\Sigma_{xy}}{\Sigma_y}(y - \bar{y})$$

$$\text{variance} = \frac{\Sigma_x \Sigma_y - \Sigma_{xy}^2}{\Sigma_y}$$

$$= \Sigma_x - \Sigma_{xy}^2 / \Sigma_y$$

For vectors,

$$E(X|Y) = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y})$$

$$\text{variance} = \Sigma_x - \underbrace{\Sigma_{xy}\Sigma_{xy}'\Sigma_y^{-1}}_{\text{will check}}$$

MMSE estimate

1. $\min_{\hat{x}} E((x-\hat{x})^2 | y) = E(X|Y)$

$$\min_{\hat{x}} E(\|x-\hat{x}\|^2 | y)$$

$$(x-\hat{x})'(x-\hat{x})$$

2. If X, Y are jointly gaussian, then

$$E(X|Y) = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y})$$

$$\text{variance} = \ldots$$

Next: apply to state-space equations in discrete time