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ELL 333

Example

x, v are independent

$v \sim \mathcal{N}(0, \Sigma_v)$

$x \sim \mathcal{N}(\bar{x}, \Sigma_x)$

$$y = Cx + v$$

What is MMSE of x given y?

MMSE → minimum mean square estimate

(or) minimum variance estimate, MVE

(y, x, v could be vectors also, but may be assumed to be scalars for simplicity)

- Last lecture: X, Y are jointly Gaussian,

$$E(X|Y) = \bar{X} + \Sigma_{xy}\Sigma_y^{-1}(Y - \bar{Y})$$

$$\Sigma_{x|y} = \Sigma_x - \underbrace{\Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}'}_{\text{noted}}$$

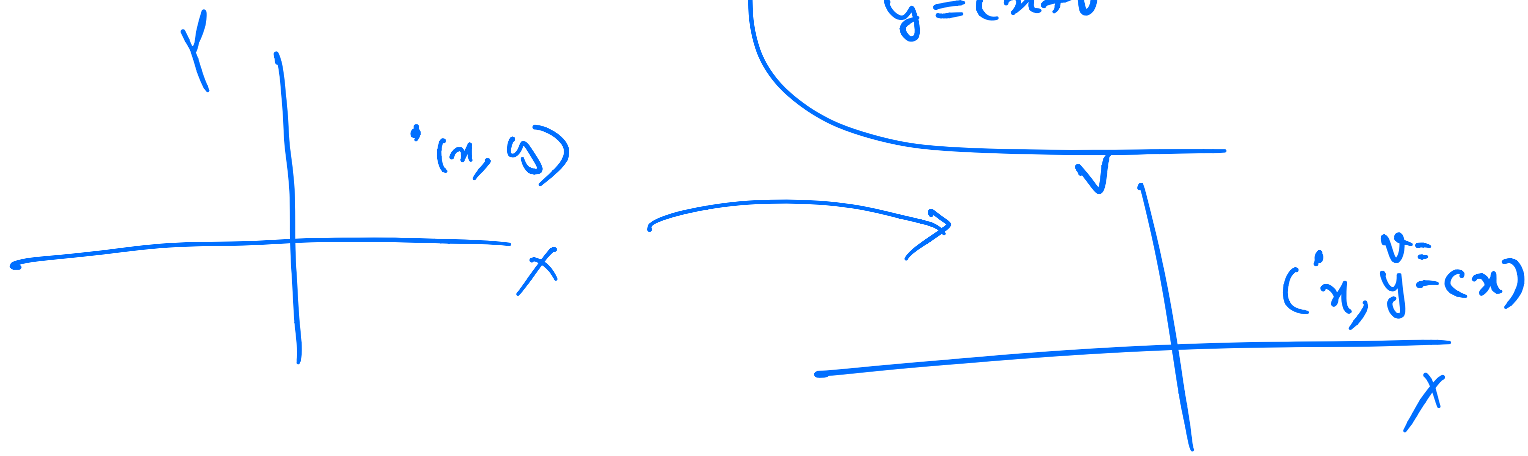
- 2nd Last lecture: $\text{MMSE} = E(X|Y)$

$\therefore$ MMSE $\rightarrow$ $E(X|Y)$ which we know how to do for jointly Gaussian X & Y.

When $y = Cx + v$, x, v are independent and gaussian,

can we say X, Y are jointly Gaussian?

$$p_{X,Y}(x,y) = \underbrace{p_{Y|X}(y)}_{y = cx + v} \cdot \underline{p_X(x)}$$



$$p_{X,Y}(x,y) = p_{X,V}(x, y - cx)$$

$$= \frac{1}{\sqrt{2\pi \Sigma_x}} \exp\left(-\frac{1}{2}\frac{(x-\bar{x})^2}{\Sigma_x}\right) \cdot \frac{1}{\sqrt{2\pi \Sigma_v}} \exp\left(-\frac{1}{2}\frac{(y-cx)^2}{\Sigma_v}\right)$$

→ Check whether it can be written in a two-dimensional Gaussian form.

$$\text{MMSE} \overset{①}{=} E(X|Y) \overset{②}{=}$$

$$\bar{X} + \Sigma_{xy}\Sigma_y^{-1}(Y - \bar{Y}),$$ $y\,x, Y$ are jointly Gaussian

$$+ \quad \boxed{y = Cx + v}$$

$$\bar{Y} = ?$$

$$\bar{Y} = E(Y) = E(CX + V)$$

$$= C\,E(X) + E(V)$$

$$= C\bar{X}$$

$$\Sigma_{xy} = E\{(X-\bar{X})(Y-\bar{Y})'\}$$

$$= E\{(X-\bar{X})(CX+V-C\bar{X})'\}$$

$$= E\{(X-\bar{X})(C(X-\bar{X})+V)'\}$$

$$= E\{(X-\bar{X})((X-\bar{X})'C'+V')\}$$

$$= E\{(X-\bar{X})(X-\bar{X})'\}C' + \underbrace{E\{(X-\bar{X})V'\}}_{=0 \text{ as } X, V \text{ are independent}}$$

$$= \Sigma_x C'$$

Similarly, calculate Σ_y?

$$\Sigma_y = E\{(Y-\bar{Y})(Y-\bar{Y})'\}$$

$$= C\Sigma_x C' + \Sigma_v$$

$$\underset{\text{MMSE}}{\hat{X}} = \bar{X} + \underbrace{\Sigma_x C'(C\Sigma_x C' + \Sigma_v)^{-1}}_{\text{gain}} \underbrace{(Y - C\bar{X})}_{\text{error}}$$

Same ideas applied to the full state space system can lead to <u>the</u> Kalman Filter.

$$x_{k+1} = A x_k + v_k$$

$$y_k = C x_k + w_k$$

Assumptions

1. $v_k \sim \mathcal{N}(0, \Sigma_v)$, $k = 0, 1, 2 \ldots$
2. $w_k \sim \mathcal{N}(0, \Sigma_w)$, $k = 0, 1, 2, \ldots$
3. $x_0 \sim \mathcal{N}(\bar{x}_0, \Sigma_{x_0})$ (initial condition)
4. $x_0, v_0, v_1, v_2 \ldots, w_0, w_1, w_2 \ldots$ are independent

independent $\overbrace{x_0, v_0}^{\text{gaussian}} \longrightarrow \overbrace{x_1}^{\text{gaussian}} = Ax_0 + v_0$

independent $\overbrace{x_1, v_1}^{\text{gaussian}} \longrightarrow \overbrace{x_2}^{\text{gaussian}} = Ax_1 + v_1$

$\cdots$

$\overbrace{x_0, w_0}^{\text{gaussi}} \longrightarrow \overbrace{y_0}^{\text{gaussi}} = Cx_0 + w_0$

gaussia $\overbrace{x_1, w_1} \longrightarrow \overbrace{y_1}^{\text{gaussia}} = Cx_1 + w_1$