ELL333
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Example:

$$y_k = C x_k + w_k$$

same assumptions as last time regarding $w_0, w_1, \ldots$ x_0 (independent Gaussian)

We have measured

$$y_0, y_1, y_2 \ldots y_{k-1} \equiv Y_{k-1}$$

and current measurement is y_k

$$(\therefore Y_k \equiv y_0, y_1, \ldots y_k)$$

expression is known when they are jointly gaussian

measurements upto k

What is $\hat{x}_{k|k} = E\{x_k | Y_k\}$

MMSE best estimate of x_k given Y_k

$$y_k = C x_k + w_k$$

idea: $x_k | Y_k = x_k | Y_{k-1} \Big| y_k | Y_{k-1}$

condition on Y_{k-1}

$$y_k | Y_{k-1} = C x_k | Y_{k-1} + w_k \Big| Y_{k-1}$$

$$\Rightarrow \underbrace{y_k | Y_{k-1}} = C \underbrace{x_k | Y_{k-1}} + w_k$$

w_k is independent

$$\Rightarrow E\left\{ x_k|Y_{k-1} \,\middle|\, y_k|Y_{k-1} \right\}$$

$$= E\{x_k|Y_{k-1}\} + \Sigma_{x_k|Y_{k-1},\, y_k|Y_{k-1}} \Sigma_{y_k|Y_{k-1}}^{-1} \left(y_k|Y_{k-1} - E\{y_k|Y_{k-1}\}\right)$$

[because if x, y are jointly gaussian,
$E\{x|y\} = E\{x\} + \Sigma_{xy}\Sigma_y^{-1}(y - E\{y\})$]

$$\Rightarrow \overset{\hat{x}_{k|k}}{E\{x_k|Y_k\}} = \overset{\hat{x}_{k|k-1}}{E\{x_k|Y_{k-1}\}} + (\quad)(\quad)\left(y_k - E\{y_k|Y_{k-1}\}\right)$$

$$E\{y_k|Y_{k-1}\} \Rightarrow C\, \underset{\hat{x}_{k|k-1}}{E\{x_k|Y_{k-1}\}}$$

$$\Sigma_{x_k|Y_{k-1},\, y_k|Y_{k-1}}$$

$$\Sigma_{y_k|Y_{k-1}}$$

$$\hookrightarrow E\left\{\left(y_k|Y_{k-1} - E\{y_k|Y_{k-1}\}\right)\left(y_k|Y_{k-1} - E\{y_k|Y_{k-1}\}\right)'\right\}$$

$$= E\left\{\left(C x_k|Y_{k-1} + w_k - C E\{x_k|Y_{k-1}\}\right)\left(C x_k|Y_{k-1} + w_k - C E\{x_k|Y_{k-1}\}\right)'\right\}$$

$$= C\, \underbrace{E\left\{\left(\overset{x_{k|k-1}}{x_k|Y_{k-1}} - \overset{\hat{x}_{k|k-1}}{E\{x_k|Y_{k-1}\}}\right)\left(\overset{x_{k|k-1}}{x_k|Y_{k-1}} - \overset{\hat{x}_{k|k-1}}{E\{x_k|Y_{k-1}\}}\right)'\right\}}_{\Sigma_{k|k-1}} C' + \Sigma_w$$

Illy, $\Sigma_{x_k|Y_{k-1},\, y_k|Y_{k-1}} = E\{(x_k|Y_{k-1} - E\{x_k|Y_{k-1}\})$

$(C(x_k|Y_{k-1} + w_k$

$- CE\{x_k|Y_{k-1}\})'\}$

$$= E\{\underbrace{(x_k|Y_{k-1} - E\{x_k|Y_{k-1}\})(x_k|Y_{k-1} - E\{x_k|Y_{k-1}\})'}_{\Sigma_{k|k-1}}\}C'$$

Simplify notation (as in green above)

$$\therefore \underbrace{\hat{x}_{k|k}}_{E\{x_k|Y_k\}} = \hat{x}_{k|k-1} + \Sigma_{k|k-1}C'(C\Sigma_{k|k-1}C' + \Sigma_w)^{-1} \times_{\text{R times}} (y_k - \underline{\hat{x}_{k|k-1}})$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - \Sigma_{k|k-1}C'(C\Sigma_{k|k-1}C' + \Sigma_w)^{-1}C\Sigma_{k|k-1}$$

[This equations comes from the variance equation]

In these equation

	RHS	+ measurement @ k	LHS
	$k\|k-1$		$k\|k$
$k=1$	$1\|0$	@ 1 $\rightarrow$	$1\|1$
$k=2$	$2\|1$	@ 2 $\rightarrow$	$2\|2$

These are "measurement update equatio"

Other equation is

$x_{k+1} = A x_k + v_k$

<u>Example</u> How to get $\hat{x}_{k+1|k}$ from $\hat{x}_{k|k}$?

$\hat{x}_{k+1|k} \to E\{x_{k+1} | Y_k\}$ $\qquad$ $\hat{x}_{k|k} \to E\{x_k | Y_k\}$

& Condition on Y_k

$\Rightarrow x_{k+1} | Y_k = A x_k | Y_k + v_k | Y_k$

$\Rightarrow x_{k+1} | Y_k = A x_k | Y_k + v_k$ why?

$\Rightarrow E\{x_{k+1} | Y_k\} = A E\{x_k | Y_k\}$

$\Rightarrow \hat{x}_{k+1|k} = A \hat{x}_{k|k}$

	<u>RHS</u>	$\longrightarrow$	<u>LHS</u>
	$k\|k$		$k+1\|k$
$k=1$	$1\|1$		$2\|1$
$k=2$	$2\|2$		$3\|2$

We also need to get $\Sigma_{k+1|k}$ from $\Sigma_{k|k}$

$$(\hat{x}_{k|k-1}, \Sigma_{k|k-1}) \longrightarrow (\hat{x}_{k|k}, \Sigma_{k|k})$$

$$(\hat{x}_{k+1|k}, \Sigma_{k+1|k}) \dashrightarrow (\hat{x}_{k+1|k+1}, \Sigma_{k+1|k+1})$$

$$\Sigma_{k+1|k} = E\{(x_{k+1|k} - \overbrace{\hat{x}_{k+1|k}}^{E\{x_{k+1}|Y_k\}})(x_{k+1|k} - \hat{x}_{k+1|k})'\}$$

$$= E\{(Ax_{k|k} + v_k - A\hat{x}_{k|k})(Ax_{k|k} + v_k - A\hat{x}_{k|k})'\}$$

$$= E\{(A(x_{k|k} - \hat{x}_{k|k}) + v_k)(A(x_{k|k} - \hat{x}_{k|k}) + v_k)'\}$$

$$= A\, E\{(x_{k|k} - \hat{x}_{k|k})(x_{k|k} - \hat{x}_{k|k})'\} A' + \Sigma_v$$

$$= A \Sigma_{k|k} A' + \Sigma_v$$

$$\left.\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} \\ \Sigma_{k+1|k} &= A\Sigma_{k|k}A' + \Sigma_v \end{aligned}\right\} \text{Time update}$$

$$\therefore \quad \hat{x}_{k|k} = \hat{x}_{k|k-1} + \Sigma_{k|k-1} C' \left(C \Sigma_{k|k-1} C' + \Sigma_w \right)^{-1} \times$$

& times

$$\left(y_k - C \underbrace{\hat{x}_{k|k-1}}_{A \hat{x}_{k-1|k-1}} \right)$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - \Sigma_{k|k-1} C' \left(C \Sigma_{k|k-1} C' + \Sigma_w \right)^{-1} C \Sigma_{k|k-1}$$