ELL333
05.11.2019

Questions

1. What is the "filter" in Kalman Filter for?
2. This Kalman Filter observer has gain L which is actually $L_k = \dfrac{2\,\Sigma_{k|k-1}}{\Sigma_{k|k-1} + \Sigma_w}$.

   Does L_k converge to some constant L? $L_k \to L$?

   Is $A - LC$ stable?
3. $L_k \to L$ if $\Sigma_{k+1|k} \longrightarrow \Sigma$. Does Σ exist?

   Note: evolution of $\Sigma_{k+1|k}$ only depends on A, C, Σ_w, Σ_v

Kalman Filter equations of the last example

$$\hat{x}_{k+1|k} = 2\,\hat{x}_{k|k-1} + \frac{2\,\Sigma_{k|k-1}}{\Sigma_{k|k-1} + \Sigma_w}\left(y_k - \hat{x}_{k|k-1}\right)$$

$$\Sigma_{k+1|k} = \frac{4\,\Sigma_{k|k-1}\,\Sigma_w}{\Sigma_{k|k-1} + \Sigma_w} + \Sigma_v$$

#3 Does $\Sigma_{k+1|k} \longrightarrow \Sigma$ as $k \to \infty$?

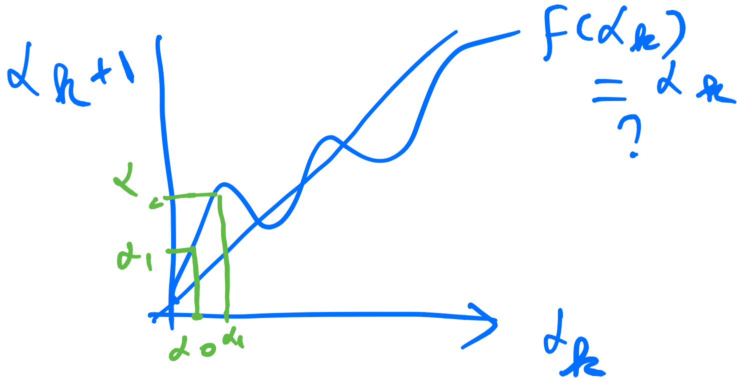
Map, f is a mapping from α_k to α_{k+1}

i.e $\alpha_{k+1} = f(\alpha_k)$, $\alpha_k \in \mathbb{R}$

We are looking for fixed points of the map

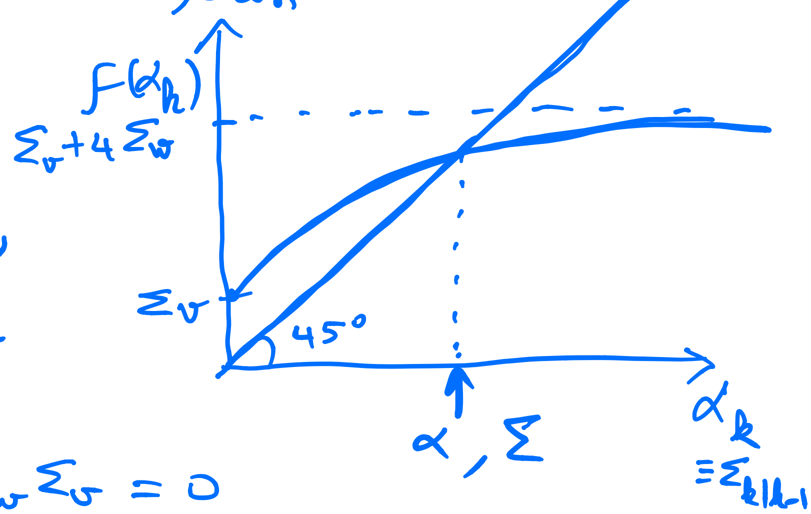
i.e $f(\alpha) = \alpha \Rightarrow \alpha$ is fixed point.

To find this, especially for scalars,



If $f(\alpha_k) = \dfrac{4\alpha_k \Sigma_w}{\alpha_k + \Sigma_w} + \Sigma_v$,

then



To find point of intersection,

$$\Sigma = \frac{4\,\Sigma\,\Sigma_w}{\Sigma + \Sigma_w} + \Sigma_v$$

$$\Sigma^2 - (3\Sigma_w + \Sigma_v)\Sigma_w - \Sigma_w\Sigma_v = 0$$

$$\Sigma = \frac{(3\Sigma_w + \Sigma_v) + \sqrt{(3\Sigma_w + \Sigma_v)^2 + 4\Sigma_w\Sigma_v}}{2}$$

This is fixed point.

How to check whether this is stable i.e

$\{\Sigma_{k|k-1}\} \longrightarrow \Sigma$?

$\alpha_{k+1} > \alpha_k$ if $\alpha_k < \alpha$

& $\alpha_{k+1} < \alpha_k$ if $\alpha_k > \alpha$

maybe next lecture

Good, cycles may be possible.

This $\Sigma_{k|k-1} \to \Sigma$ is important.

#2 Because, if so, then $L_k \longrightarrow L = \dfrac{2\Sigma}{\Sigma + \Sigma_w}$

Then we can exactly determine the dynamics of the observer-like equation,

$$\hat{x}_{k+1|k} = 2\hat{x}_{k|k-1} + \underbrace{\frac{2\,\Sigma_{k|k-1}}{\Sigma_{k|k-1} + \Sigma_w}}_{\to 2L_k}\left(y_k - \hat{x}_{k|k-1}\right)$$

$$x_{k+1} = 2x_k + v_k, \quad y_k = x_k + w_k$$

$$e_{k+1} = x_{k+1} - \underline{\underline{\hat{x}_{k+1|k}}}$$

$$= 2x_k + v_k$$

$$- 2\hat{x}_{k|k-1} - \underline{\underline{L_k}}\left(x_k + w_k - \hat{x}_{k|k-1}\right)$$

$$= 2e_k + v_k - L_k e_k + L_k w_k$$

$$= (2 - L_k)\,e_k + v_k + L_k w_k$$

$$e_{k+1} = (2 - L_k)\,e_k + v_k + L_k w_k$$

$$2 - L_k = 2 - 2\frac{\Sigma_{k|k-1}}{\Sigma_{k|k-1} + \Sigma_w} = 2 \cdot \frac{\Sigma_w}{\Sigma_{k|k-1} + \Sigma_w}$$

Suppose $\Sigma_{k|k-1} \to \Sigma$, $\Rightarrow$

$$2 \cdot \frac{\Sigma_w}{\Sigma + \Sigma_w}$$

$$\Sigma = \frac{(3\Sigma_w + \Sigma_v) + \sqrt{(3\Sigma_w + \Sigma_v)^2 + 4\Sigma_w \Sigma_v}}{2}$$

$$2 - L = 2 \frac{\Sigma_w}{\Sigma + \Sigma_w}$$

Suppose $\Sigma_w = \Sigma_v = 1$, $\quad \Sigma = \frac{4 + \sqrt{20}}{2}$

$$= 2 + \sqrt{5}$$

$$\Rightarrow 2 - L = 2 \frac{1}{3 + \sqrt{5}} < 1$$

In this case, $\quad \Sigma > \Sigma_w \Rightarrow \Sigma + \Sigma_w > 2\Sigma_w$

$$\Rightarrow \frac{1}{\Sigma + \Sigma_w} < \frac{1}{2\Sigma_w}$$

$$\Rightarrow \frac{2\Sigma_w}{\Sigma + \Sigma_w} < 1$$

(mean)

$\Rightarrow$ error dynamics converge