ELL333

13.11.2019

1. $m\ddot{x} + kx = u(t)$

↳ common model for flexible structures (aeroplane wing, ships, tall buildings)

Can we use input 'u' to make the deflection $x(t)$ and its velocity $\dot{x}(t)$ go to zero in finite time from arbitrary initial conditions?

≡ controllability.

Is the system controllable?

→ State space model

$$x_1 = x, \quad x_2 = \dot{x}$$

$$\Rightarrow \dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m} x_1 + \frac{1}{m} u(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$\text{rank}[B \;\; AB] = \text{rank}\begin{bmatrix} 0 & 1/m \\ 1/m & 0 \end{bmatrix} = 2$$

2. Transfer functions $\longleftrightarrow$ Jordan canonical form.

$$\frac{Y(s)}{U(s)} = \frac{1}{(s+2)(s+3)}$$

Convert this into state-space?

$$\Rightarrow (s^2 + 5s + 6) Y(s) = U(s)$$

$$\Longleftrightarrow \quad \ddot{y} + 5\dot{y} + 6y = u$$

$$x_1 = y, \quad x_2 = \dot{y}$$

$$\Rightarrow \quad \dot{x}_1 = x_2$$

$$\dot{x}_2 = -6x_1 - 5x_2 + u$$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Another way: partial fractions

$$\frac{Y(s)}{U(s)} = \frac{1}{(s+2)(s+3)} = \underbrace{\frac{1}{s+2}}_{\dot{x}_1 = -2x_1 + u} - \underbrace{\frac{1}{s+3}}_{\dot{x}_2 = -3x_2 + u}$$

$$y = x_1 - x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overset{A}{\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overset{B}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} u$$

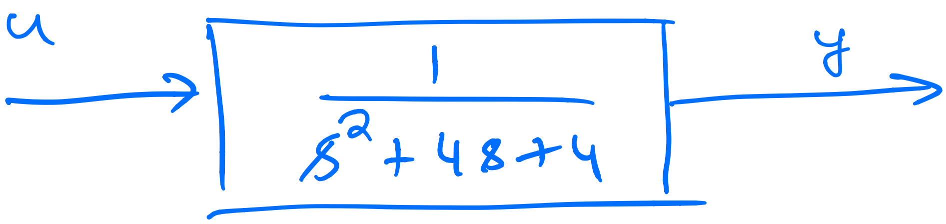
$$y = \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

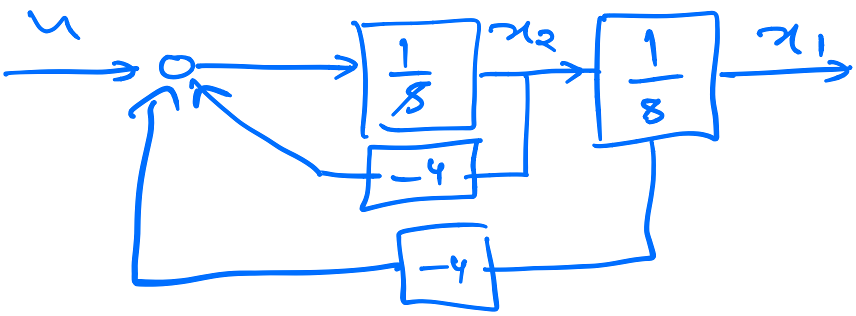
Repeat for $\dfrac{Y(s)}{U(s)} = \dfrac{1}{(s+2)^2}$

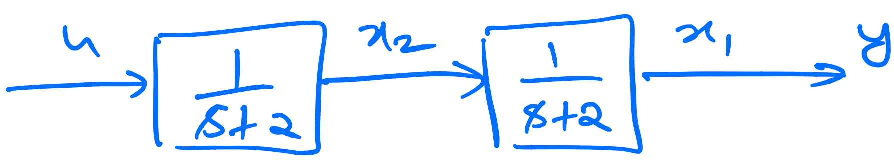
$$A = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}}_{\text{Companion form.}}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

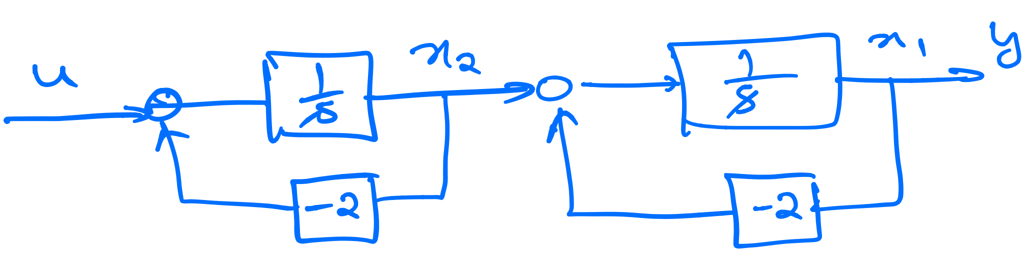
$\dot{x}_1 = x_2$

$\dot{x}_2 = -4x_1 - 4x_2 + u$









$\dot{x}_2 = -2x_2 + u$ $\qquad$ $\dot{x}_1 = -2x_1 + x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Eigenvectors for $A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

eigenvalues are $-2, -2$

$A - \lambda I, \quad \lambda = -2$

$$A + 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for generalized eigenvectors, we have to solve $\underbrace{(A - \lambda I)^2}\, v_i = 0$

choose

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} v_i = 0, \quad \Rightarrow v_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

doesn't help to understand generalized eigenvectors

Lyapunov functions

Generalized energy.

for 2D $\alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$

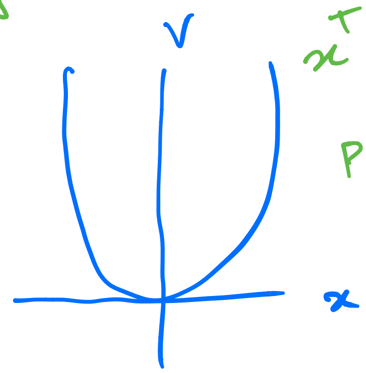
$x^T P x$

$$P = \begin{bmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{bmatrix}$$

$$\dot{x} = -x$$

$$V(x) = \frac{x^2}{2}$$

Lyapunov function



Time derivative along system trajectories

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \dot{x}$$

$$= x(-x)$$

$$= -x^2 < 0 \quad \forall x \neq 0$$

$$V(x) = x^T P x \quad , \dot{x} = Ax$$

$$\frac{dV}{dt} = x^T P \dot{x} + \dot{x}^T P x$$

$$= x^T P A x + x^T A^T P x$$

$$= x^T (PA + A^T P) x$$