ELL 333 MAJOR TEST DURATION = 2 HR, MARKS = 36

1. Suppose the $n \times n$ matrix

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & 0 \\ -a_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 0 & 1 \\ -a_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$. Find the invertible matrix T that diagonalizes A. $T^{-1}AT = D = \mathrm{diag}\{\lambda_1, \lambda_2 \ldots \lambda_n\}$. $T = ?$ <u>4 marks</u>

$T = [v_1 \; v_2 \; \ldots \; v_n]$ where v_i is eigenvector corresponding to λ_i. 1

$$\lambda_i I - A = \begin{bmatrix} \lambda_i + a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_2 & \lambda_i & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 0 & 0 & \cdots & \lambda_i & -1 \\ a_n & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$ 1

$$\sim \begin{bmatrix} \lambda_i + a_1 & -1 & 0 & \cdots & 0 & 0 \\ \lambda_i^2 + a_1\lambda_i + a_2 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_i^{n-1} + \cdots + a_{n-1} & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$ 2

$$\therefore \; v_i = \begin{bmatrix} 1 \\ \lambda_i + a_1 \\ \lambda_i^2 + a_1\lambda_i + a_2 \\ \vdots \\ \lambda_i^{n-1} + a_1\lambda_i^{n-2} + \cdots + a_{n-1} \end{bmatrix}$$

2. X and Y are scalar random variables with joint density $f_{XY}(x,y) = \frac{1}{\sqrt{(2\pi)^2 (\det \Sigma)}} \exp\left(\frac{-1}{2}\begin{bmatrix} x-\bar{x} \\ y-\bar{y}\end{bmatrix}' \Sigma^{-1} \begin{bmatrix} x-\bar{x} \\ y-\bar{y}\end{bmatrix}\right)$,

$\Sigma = \begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_y \end{bmatrix}$, $\Sigma_{xy} = \Sigma_{yx}$. Find the mean and variance of $X|Y$. <u>4 marks</u>

$$\Sigma^{-1} = \frac{1}{\Sigma_x\Sigma_y - \Sigma_{xy}\Sigma_{yx}} \begin{bmatrix} \Sigma_y & -\Sigma_{xy} \\ -\Sigma_{yx} & \Sigma_x \end{bmatrix} \quad 1$$

$$f_{X|Y}(x) = \frac{f_{XY}(x,y)}{\int_x f_{XY}(x,y)\,dx} \quad 1$$

$$\Rightarrow \frac{\Sigma_y}{\Sigma_x\Sigma_y - \Sigma_{xy}\Sigma_{yx}} \left[(x-\bar{x}) - \frac{\Sigma_{xy}}{\Sigma_y}(y-\bar{y})\right]^2$$

+ other terms which cancel

$$f_{X|Y}(x) = \frac{1}{\sqrt{2\pi\left(\Sigma_x - \frac{\Sigma_{xy}^2}{\Sigma_y}\right)}} \exp\left\{-\frac{1}{2} \frac{\left[(x-\bar{x}) - \frac{\Sigma_{xy}}{\Sigma_y}(y-\bar{y})\right]^2}{\Sigma_x - \Sigma_{xy}^2/\Sigma_y}\right\} \quad 2$$

$$\text{mean} = \bar{x} + \frac{\Sigma_{xy}}{\Sigma_y}(y-\bar{y})$$

$$\text{variance} = \Sigma_x - \Sigma_{xy}^2/\Sigma_y$$

3. Consider the discrete-time system

$x_{k+1} = Ax_k + v_k$, $v_k, x_k \in \mathbb{R}^2$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$y_k = Cx_k + w_k$, $y_k, w_k \in \mathbb{R}$, $C = [1 \ \ 0]$,

with usual assumptions : $E\{v_k\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $E\{w_k\} = 0$, $E\{w_k w_k'\} = \Sigma_w$, $E\{v_k v_k'\} = \Sigma_v = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$, v and w are independent and Gaussian white noise processes.

a) Under what conditions $\text{rank}\left\{\begin{bmatrix} C \\ CA \end{bmatrix}\right\} = 2$?

b) Derive the Kalman filter in terms of given matrices

c) Denote $\Sigma_{k|k-1} = \begin{bmatrix} \alpha_k & \beta_k \\ \beta_k & \gamma_k \end{bmatrix}$. How do $\alpha_k, \beta_k, \gamma_k$ evolve as k increases for $a_{11} = 1 = a_{22}$, $a_{12} = 0 = a_{21}$? Analyse their convergence. $1 + 5 + 4 = 10$ marks

a) $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_{11} & a_{12} \end{bmatrix}$ 1

rank $= 2$ if $a_{12} \neq 0$

b) measurement update

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \frac{\Sigma_{k|k-1} C'}{C \Sigma_{k|k-1} C' + \Sigma_w} (y_k - C \hat{x}_{k|k-1})$$

follows by considering covariances

$$\Sigma_{k|k} = \Sigma_{k|k-1} - \frac{\Sigma_{k|k-1} C' C \Sigma_{k|k-1}}{C \Sigma_{k|k-1} C' + \Sigma_w}$$ 3

This is from the MMSE estimate

time update

$$\hat{x}_{k+1|k} = A \hat{x}_{k|k}$$

$$\Sigma_{k+1|k} = A \Sigma_{k|k} A' + \Sigma_v$$

2

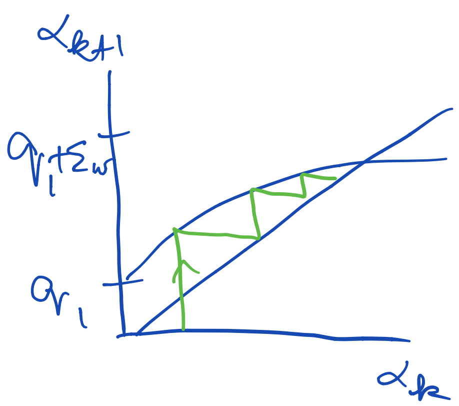
This follows directly from taking the covariances.

c) $$\alpha_{k+1} = \frac{\alpha_k \Sigma_w}{\alpha_k + \Sigma_w} + q_1$$ on plugging in values

$$\beta_{k+1} = \frac{\beta_k \Sigma_w}{\alpha_k + \Sigma_w}$$

2

$$\gamma_{k+1} = \gamma_k - \frac{\beta_k^2}{\alpha_k + \Sigma_w} + q_2$$



$$\beta_k = \Big(\underbrace{\frac{\Sigma_w}{\alpha_k + \Sigma_w}}_{<1}\Big)^k \beta_0 \longrightarrow 0$$

α_k converges

β_k converges

2

$$\gamma_k \cong r_0 + k q_2 \quad \text{for large } k$$

$\Rightarrow \gamma_k$ doesn't converge

4. "A paragraph is a group of related sentences that develop a single point." Write a paragraph on the Kalman filter on a point of your choice related to it. <u>4 marks</u>

Examples:

1 mark → a sentence

The Kalman Filter is an optimal observer for a linear system with White Gaussian noise.

2 marks → multiple sentences

The Kalman Filter is an optimal observer. It has many applications in navigation. It does not work in nonlinear cases.

3 marks → multiple sentences aiming towards a point

The Kalman Filter is an optimal observer. It is optimal for linear systems with white Gaussian noise. It has two steps: time update and measurement update. The gain can be updated at each time step.

4 marks → multiple sentences that make a coherent point

The Kalman Filter is an algorithm to estimate states of a system given noisy output measurements. For a linear system, continuous, discrete or continuous-discrete, in the presence of Gaussian white noise, it is the optimal estimate. The core of the algorithm has two steps: One, a prediction step that predicts the future value of the state and its variance based on the system equations. Because of process noise, this step typically increases the variance. Two, a correction step that updates the state estimate and its variance based on the minimum mean square estimate (MMSE) theory. As the output is measured, this typically reduces the variance. These steps lend themselves easily to a computer implementation and are an impor-tant source of its popularity, with multiple applications throughout engineering and science.

5. A factor determining useful life of a flexible structure, such as a ship, a tall building, or a large aeroplane, is the possibility of fatigue failures due to structural vibrations. Each vibration mode is described by an equation of the form $m\ddot{x} + kx = u(t)$, where $u(t)$ is the input force. Is it possible to find an input which will drive both the deflection $x(t)$ and the velocity $\dot{x}(t)$ to zero in finite time for arbitrary initial conditions?

<u>3 marks</u>

need to check controllability 1

$x_1 = x, \; x_2 = \dot{x}$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u$$ 1

$$\text{rank}\{[B \;\; AB]\} = \text{rank}\left\{\begin{bmatrix} 0 & 1/m \\ 1/m & 0 \end{bmatrix}\right\} = 2$$ 1

<u>Yes</u>.

6. Consider the system

$$\dot{x} = Ax + Bu, \quad A = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ -4 & 5 & 0 \\ \hline 0 & 0 & 4 \end{array}\right], \quad B = \left[\begin{array}{c} 1 \\ 1 \\ \hline 0 \end{array}\right].$$

Answer the following with clear justification.

a) Is it stable?

b) Is it controllable?

c) Does there exist a function $u(t)$ so that $x(1) = \begin{bmatrix} 11 \\ 12 \\ 1 \end{bmatrix}$ starting from $x(0) = \begin{bmatrix} e^{-3} \\ 2e^{-3} \\ e^{-4} \end{bmatrix}$?

<u>1 + 2 + 4 = 7 marks</u>

a) No. one eigenvalue is $+4$.

For other eigenvalues solve $\det\left\{\lambda I - \begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}\right\} = 0$

They are $+1, +3$. 1

b) $\text{rank}\{[B \;\; AB \;\; A^2B]\}$

$= \text{rank} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 1 < 3$, not controllable

2

c) Yes.

Note that $x_3(t) = x_3(0)\, e^{4t} \Rightarrow x_3(1) = 1$ for any $u(t)$ and given initial condition 1

For other two states

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{Z}} = \underbrace{\begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{Z} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{F} u$$

$$\Rightarrow z(t) = e^{Et} z(0) + \int_0^t e^{E(t-\tau)} F u(\tau)\, d\tau$$

$$\Rightarrow z(t) - e^{Et} z(0) = \underbrace{\int_0^t e^{E(t-\tau)} F u(\tau)\, d\tau}$$

range of $[F \; EF \ldots E^{n-1}F]$
by definition of matrix
exponential and
Cayley Hamilton theorem

Check $z(1) - e^{E1} z(0)$

$$e^{E1} = \begin{bmatrix} -e^3+2e & e^3-e \\ -2e^3+2e & 2e^3-e \end{bmatrix}$$

$\mathrm{eig}(E) = 3, 1$

eigenvectors $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \; T^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

$T^{-1}ET = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = D$

$\Rightarrow e^{Et} = T e^{Dt} T^{-1}$

$z(1) - e^{E1} z(0)$

$$= \begin{bmatrix} 11 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$ is in the

range of $[F \;\; EF] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

7. A model of a mobile robot is $\dot{q} = u$, where q is its position. Consider N mobile robots numbered 1 to N. Let q_i denote the position of robot i. One way to design a control law for the robots to meet is a cyclic pursuit: Suppose robot N pursues robot N−1 as per the equation

$$\dot{q}_N = q_{N-1} - q_N$$

and so on down to

$$\dot{q}_2 = q_1 - q_2$$
$$\dot{q}_1 = q_N - q_1 .$$

a) Show that the centroid of robots $q_c = \frac{1}{N}\sum_{i=1}^{N} q_i$ is fixed at its initial position, $q_c(t) = q_c(0)$.

b) For $N = 2$ show that both robots approach $q_c(0)$ as $t \to \infty$.

<u>1 + 3 = 4 marks</u>

a) $$\dot{q}_c = \frac{1}{N}\sum_{i=1}^{N} \dot{q}_i = 0$$

$$\Rightarrow q_c(t) = q_c(0)$$

b) $$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \exp(At) \begin{bmatrix} q_1(0) \\ q_2(0) \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda+1 & -1 \\ -1 & \lambda+1 \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda+1)^2 - 1 = \lambda(\lambda+2)$$

eigenvalues : $0, -2$

eigenvectors : $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then, $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $T^{-1} = \frac{-1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

and $T^{-1} A T = D = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$

$$\Rightarrow \exp(At) = T \exp(Dt) T^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \begin{bmatrix} 1 + e^{-2t} & 1 - e^{-2t} \\ 1 - e^{-2t} & 1 + e^{-2t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (1+e^{-2t}) q_1(0) + (1-e^{-2t}) q_2(0) \\ (1-e^{-2t}) q_1(0) + (1+e^{-2t}) q_2(0) \end{bmatrix} \rightarrow \begin{bmatrix} q_c(0) \\ q_c(0) \end{bmatrix}$$