ELL 333 MINOR TEST 1 MARKS = 20 DURATION = 1 HOUR

SOLUTIONS

1. For a bicycle model, the linearized equations are $M\ddot{q} + v C_1 \dot{q} + [gK_0 + v^2 K_2] q = f$, where $q = [\phi, \delta]'$, $f = [T_\phi, T_\delta]'$, $M = \begin{bmatrix} 80.8 & 2.3 \\ 2.3 & 0.3 \end{bmatrix}$, $g = 9.8$, $v = 2$, $C_1 = \begin{bmatrix} 0 & 33.9 \\ -0.9 & 1.7 \end{bmatrix}$, $K_0 = \begin{bmatrix} -81 & -2.6 \\ -2.6 & -0.8 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0 & 76.6 \\ 0 & 2.7 \end{bmatrix}$.

(T_ϕ is lean torque, T_δ is steering torque)

a) Using $x = [\phi \ \delta \ \dot{\phi} \ \dot{\delta}]'$, $u = f$, write the state-space model $\dot{x} = Ax + Bu$. $A = ?$, $B = ?$

b) Is the system controllable using the input T_ϕ only ($T_\delta \equiv 0$, it is not there).

c) Is the system controllable using the input T_δ only ($T_\phi \equiv 0$, it is not there).

a) $x_1 = \phi$, $x_2 = \delta$, $x_3 = \dot{\phi}$, $x_4 = \dot{\delta}$

$\Rightarrow \dot{x}_1 = x_3$, $\dot{x}_2 = x_4$

$$M\ddot{q} + v C_1 \dot{q} + [gK_0 + v^2 K_2] q = f$$

$$\Rightarrow M\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + v C_1 \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + [gK_0 + v^2 K_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = u$$

$$\Rightarrow M\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = -[gK_0 + v^2 K_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - v C_1 \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + u$$

$$\Rightarrow \begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = -M^{-1}[gK_0 + v^2 K_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - M^{-1} v C_1 \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + M^{-1} u$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -M^{-1} \times [gK_0 + v^2 K_2] & & -M^{-1} \times vC_1 & \end{array}\right]}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline \multicolumn{2}{c}{M^{-1}} \end{array}\right]}_{B} u$$

Numerical values are, $M = \begin{bmatrix} 80.8 & 2.3 \\ 2.3 & 0.3 \end{bmatrix}$, $g = 9.8$,

$v = 2$, $C_1 = \begin{bmatrix} 0 & 33.9 \\ -0.9 & 1.7 \end{bmatrix}$, $K_0 = \begin{bmatrix} -81 & -2.6 \\ -2.6 & -0.8 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0 & 76.6 \\ 0 & 2.7 \end{bmatrix}$.

$$M^{-1} = \frac{1}{\underset{24.24}{80.8 \times 0.3} - \underset{5.29}{(2.3)^2}} \begin{bmatrix} 0.3 & -2.3 \\ -2.3 & 80.8 \end{bmatrix}$$

$$= 0.053 \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} 0.016 & -0.122 \\ -0.122 & 4.28 \end{bmatrix}$$

$$M^{-1} v C_1 = 2 \begin{bmatrix} 0.016 & -0.122 \\ -0.122 & 4.28 \end{bmatrix} \begin{bmatrix} 0 & 33.9 \\ -0.9 & 1.7 \end{bmatrix}$$

$$= \begin{bmatrix} 2(0.122 \times 0.9) & 2(\underset{0.542}{0.016 \times 33.9} - \underset{0.207}{0.122 \times 1.7}) \\ -2 \times 4.28 \times 0.9 & -2(\underset{4.14}{33.9 \times .122} - \underset{7.28}{4.28 \times 1.7}) \end{bmatrix}$$

$$= \begin{bmatrix} 0.22 & 0.67 \\ -7.7 & 6.28 \end{bmatrix}$$

$$M^{-1}\left[gK_0 + v^2 K_2\right] = M^{-1}\left(\begin{bmatrix} -81\times 9.8 & \overset{25.48}{-2.6\times 9.8} \\ -2.6\times 9.8 & \overset{7.84}{-0.8\times 9.8} \end{bmatrix} + \begin{bmatrix} 0 & \overset{306.4}{4\times 76.6} \\ 0 & \overset{10.8}{4\times 2.7} \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0.016 & -0.122 \\ -0.122 & 4.28 \end{bmatrix}\begin{bmatrix} -793.8 & 280.92 \\ -25.48 & 2.96 \end{bmatrix}$$

$$= \begin{bmatrix} -\underset{12.7}{0.016\times 793.8} + \underset{3.12}{0.122\times 25.48} & \overset{4.49}{280.92\times 0.016} - \underset{0.36}{0.122\times 2.96} \\ \underset{96.8}{0.122\times 793.8} - \underset{109.05}{4.28\times 25.48} & -\overset{34.27}{280.92\times 0.122} + \underset{12.67}{4.28\times 2.96} \end{bmatrix}$$

$$= \begin{bmatrix} -9.58 & 4.13 \\ -12.25 & -21.6 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.58 & -4.13 & -0.22 & -0.67 \\ 12.25 & 21.6 & 7.7 & -6.28 \end{bmatrix}$$

$$B = \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0.016 & -0.122 \\ -0.122 & 4.28 \end{array}\right] \end{array}$$

b) & c) check

rank $[b_i \;\; Ab_i \;\; A^2 b_i \;\; A^3 b_i] = 4$

$i=1$ for b)

$i=2$ for c)

2. A and B are square matrices.

a) Show that, if $AB = BA$, $e^{A+B} = e^A \cdot e^B$.

b) By counterexample, choose A and B such that $AB \neq BA$ and check that $e^{A+B} \neq e^A \cdot e^B$

a) $$e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \frac{(A+B)^3}{3!} + \dots$$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \dots$$

$$e^A \cdot e^B = I + (A+B) + \left(\frac{A^2}{2!} + \frac{B^2}{2!} + AB\right) + \dots$$

Now, $(A+B)^2 = (A+B)(A+B)$

$$= A^2 + AB + BA + B^2$$

If $AB = BA$, then $(A+B)^2 = A^2 + 2AB + B^2$

and $e^{A+B} = e^A \cdot e^B$

(similarly for higher powers of $(A+B)$)

b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Here $AB \neq BA$.

$$A+B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^A = I + A \qquad \left[\because A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right.$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^B = I + B + \frac{B^2}{2!} + \cdots \qquad \left[B^2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right.$$

$$= I + B\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) \qquad = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = B$$

$$= I + B(e-1)$$

$$= \begin{bmatrix} 1 & 1-e \\ 0 & e \end{bmatrix}$$

$$e^A \cdot e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1-e \\ 0 & e \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & e \end{bmatrix}$$

$$e^{A+B} = I + (A+B)(e-1) \qquad \left[\because (A+B)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right.$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} \qquad = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3. A disastrous way to control an unstable system is to cancel the unstable pole with a zero. Consider a transfer function from input to output as $H(s) = \frac{1}{s^2 - \Omega^2}$

a) Is this unstable?

Consider a controller $G(s) = \frac{s - \bar{\Omega}}{s}$.

Clearly, $G(s)H(s) = \frac{1}{s(s+\Omega)}$ when $\Omega \overset{s}{=} \bar{\Omega}$.

When $\Omega \neq \bar{\Omega}$, the transfer function $G(s)H(s)$ can be shown to be in the form $\dot{x} = Ax + Bu$, where
$$A = \begin{bmatrix} 0 & 1 & 0 \\ \Omega^2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ \bar{\Omega} \end{bmatrix}.$$

b) Find the transformation T that diagonalizes A. $AT = TD$

c) Express the equation in z-co.ordinates, where $z = T^{-1}x$. $(z = [z_1 \; z_2 \; z_3]')$

d) Analyze the z-equations in the limit $\bar{\Omega} \to \Omega$. Can they be controlled?

a) Unstable as pole $= +1$ in RHP

b) eigenvalues of A,
$$sI - A = \begin{bmatrix} s & -1 & 0 \\ -\Omega^2 & s & 1 \\ 0 & 0 & s \end{bmatrix}$$

$$\det(sI-A) = s(s^2+\Omega^2)$$

$\therefore$ eigenvalues are $\Omega, 0, -\Omega$

eigenvectors:

$\underline{s=\Omega}$, $sI-A = \begin{bmatrix} \Omega & -1 & 0 \\ -\Omega^2 & \Omega & 1 \\ 0 & 0 & \Omega \end{bmatrix}$

$\sim \begin{bmatrix} \Omega & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \Omega \end{bmatrix} \sim \begin{bmatrix} \Omega & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ eigenvector can be $\begin{bmatrix} 1 \\ \Omega \\ 0 \end{bmatrix}$

$\underline{s=0}$, $sI-A = \begin{bmatrix} 0 & -1 & 0 \\ -\Omega^2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ eigenvector can be $\begin{bmatrix} 1 \\ 0 \\ \Omega^2 \end{bmatrix}$

$\underline{s=-\Omega}$, $sI-A = \begin{bmatrix} -\Omega & -1 & 0 \\ -\Omega^2 & -\Omega & 1 \\ 0 & 0 & -\Omega \end{bmatrix}$

$$\sim \begin{bmatrix} -\Omega & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\Omega \end{bmatrix} \sim \begin{bmatrix} \Omega & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ eigenvector can be $\begin{bmatrix} 1 \\ -\Omega \\ 0 \end{bmatrix}$

The transformation matrix, $T = \begin{bmatrix} 1 & 1 & 1 \\ \Omega & 0 & -\Omega \\ 0 & \Omega^2 & 0 \end{bmatrix}$

c) $$\dot{z} = T^{-1}\dot{x}$$
$$= \underbrace{T^{-1}AT}_{\begin{bmatrix} \Omega & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Omega \end{bmatrix}} z + T^{-1}Bu$$

$$T^{-1} = \frac{\text{adj}(T)}{\det(T)}$$

$$\det(T) = -\Omega^2 \begin{vmatrix} 1 & 1 \\ \Omega & -\Omega \end{vmatrix} = 2\Omega^3$$

$$\text{adj}(T) = \begin{bmatrix} \Omega^3 & 0 & \Omega^3 \\ \Omega^2 & 0 & -\Omega^2 \\ -\Omega & 2\Omega & -\Omega \end{bmatrix}^T = \begin{bmatrix} \Omega^3 & \Omega^2 & -\Omega \\ 0 & 0 & 2\Omega \\ \Omega^3 & -\Omega^2 & -\Omega \end{bmatrix}$$

$$\therefore T^{-1}B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\Omega} & -\frac{1}{2\Omega^2} \\ 0 & 0 & \frac{1}{\Omega^2} \\ \frac{1}{2} & -\frac{1}{2\Omega} & -\frac{1}{\Omega^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \bar{\Omega} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\Omega^2}(\Omega - \bar{\Omega}) \\ \frac{\bar{\Omega}}{\Omega^2} \\ \frac{-1}{2\Omega^2}(\Omega + \bar{\Omega}) \end{bmatrix}$$

as $\bar{\Omega} \to \Omega$,

$$\dot{z}_1 = \Omega z_1 + \frac{1}{2\Omega^2}\overbrace{(\Omega - \bar{\Omega})}^{\to 0 \text{ pole-zero cancellation}} u$$

$$\dot{z}_2 = \frac{\bar{\Omega}}{\Omega^2} u$$

$$\dot{z}_3 = -\Omega z_3 - \frac{1}{2\Omega^2}(\Omega + \bar{\Omega}) u$$

z_1 - equation is unstable and there is no input term