ELL 333 MINOR TEST 2 MARKS = 20 DURATION = 1 HOUR

SOLUTIONS

1. For a bicycle model, the state space equations are $\dot{x} = Ax + Bu$, $y = Cx$, where $x = \begin{bmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{bmatrix}$

and $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10 & -4 & 0 & -1 \\ 12 & 22 & 8 & -6 \end{bmatrix}$.

Is the system observable if

a) only ϕ is measured?

b) only δ is measured?

c) only $\dot{\phi}$ is measured?

d) only $\phi - \delta$ is measured?

6 marks

Clearly & briefly justify all steps.

Observable if $\text{rank}\left\{\underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}}_{W_o}\right\} = 4$ for different C matrices

$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10 & -4 & 0 & -1 \\ 12 & 22 & 8 & -6 \end{bmatrix}$, $A^2 = \begin{bmatrix} 10 & -4 & 0 & -1 \\ 12 & 22 & 8 & -6 \\ -12 & -22 & 2 & 2 \\ 8 & -164 & -36 & 50 \end{bmatrix}$

$A^3 = \begin{bmatrix} -12 & -22 & 2 & 2 \\ 8 & -164 & -36 & 50 \\ 44 & 36 & 4 & -36 \\ 240 & 1244 & 408 & -428 \end{bmatrix}$

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a) $C = [1 \quad 0 \quad 0 \quad 0]$

$$W_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 10 & -4 & 0 & -1 \\ -12 & -22 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & 0 & -1 \\ 0 & -22 & 0 & 2 \end{bmatrix}$$

$\Rightarrow$ rank $= 4 \Rightarrow$ observable. 1

b) $C = [0 \quad 1 \quad 0 \quad 0]$

$$W_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 12 & 22 & 8 & -6 \\ 8 & -164 & -36 & 50 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 12 & 0 & 8 & 0 \\ 8 & 0 & -36 & 0 \end{bmatrix}$$

$\Rightarrow$ rank $= 4 \Rightarrow$ observable. 1

c) $C = [0 \quad 0 \quad 1 \quad 0]$

$$W_o = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 10 & -4 & 0 & -1 \\ -12 & -22 & 2 & 2 \\ 44 & 36 & 4 & -36 \end{bmatrix}$$

$$\det(W_o) = \begin{vmatrix} 10 & -4 & -1 \\ -12 & -22 & 2 \\ 44 & 36 & -36 \end{vmatrix} = \begin{vmatrix} 10 & -4 & -1 \\ 8 & -30 & 0 \\ -316 & -108 & 0 \end{vmatrix}$$

$\neq 0 \Rightarrow$ rank $= 4 \Rightarrow$ observable. 1

d) $C = [1 \quad -1 \quad 0 \quad 0]$

$$W_o = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -2 & -26 & -8 & 5 \\ -20 & 142 & 38 & -48 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -28 & -8 & -3 \\ -20 & 122 & 38 & -10 \end{bmatrix}$$

$\Rightarrow$ rank $= 4$

$\Rightarrow$ observable

1

2. Show that $\boxed{4+2 \text{ marks}}$

a) (A, B) is controllable $\Rightarrow$ $\text{rank}\{[B \; AB \; \dots \; A^{n-1}B]\} = n$

$\{$ Hint: Controllability Grammian,

$$P(T) = \int_0^T e^{A(T-t)} B B' e^{A'(T-t)} dt \quad \}$$

b) $\text{Ker}(Ce^{At}) = \text{Ker}(C) \cap \text{Ker}(CA) \dots \cap \text{Ker}(CA^{n-1})$

Suppose $P(T)$ is singular

$\Rightarrow \exists\, v (\neq 0)$ such that

$$v' P v = 0$$

$$\Rightarrow \quad B' e^{A'(T-t)} v = 0$$

$$\& \quad v' e^{A(T-t)} B = 0$$

Can state v be reached?

should be as (A, B) is controllable.

$\therefore \exists\, u$ such that

$$v = \int_0^T e^{A(T-t)} B u(t)\, dt$$

Pre-multiply by v' $\Rightarrow$ $\underbrace{v'v}_{\neq 0} = \underbrace{\int_0^T v' e^{A(T-t)} B u(t) dt}_{= 0}$

contradiction

$\therefore$ $P(T)$ is non-singular 2

Suppose $\text{rank}\{[B \; AB \dots A^{n-1}B]\} < n$

$\Rightarrow \exists\, v (\neq 0)$ such that

$$v'[B \;\; AB \dots A^{n-1}B] = 0$$

$$\Rightarrow v'B = v'AB = \dots = v'A^{n-1}B = 0$$

Now $e^{A(T-t)}$, via the Cayley Hamilton Thm can be expressed as a combination of $\{I, A, \dots A^{n-1}\}$

$\Rightarrow e^{A(T-t)}B = f_0(T-t)B + f_1(T-t)AB + \dots f_{n-1}(T-t)A^{n-1}B$

$\Rightarrow v' e^{A(T-t)}B = 0$

$\Rightarrow v' P(T) = 0$

$\Rightarrow P(T)$ is singular.

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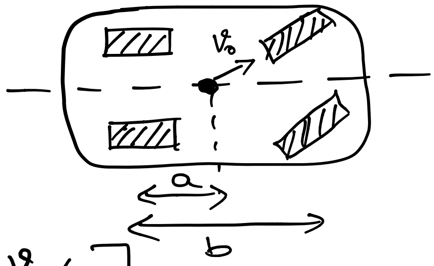
b) From class notes

show $x \in LHS \Rightarrow x \in RHS$ 2

& $x \in RHS \Rightarrow x \in LHS$

3. A linearized model of vehicle steering (no slip) is



$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$v_0 > 0$

$$A = \begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} a v_0/b \\ v_0/b \end{bmatrix}, \quad C = [1 \quad 0].$$

a) Show that (A, B) is controllable. Design a controller $u = -Kx$ to place closed loop eigenvalues at the roots of the equation $s^2 + 2\zeta_c \omega_c s + \omega_c^2 = 0$. $\zeta_c, \omega_c > 0$

b) Show that (A, C) is observable. Design an observer with gain L so that the eigenvalues of $A - LC$ are at the roots of the equation $s^2 + 2\zeta_o \omega_o s + \omega_o^2 = 0$. $\zeta_o, \omega_o > 0$

c) What factors might govern the choice of $\{\zeta_c, \omega_c, \zeta_o, \omega_o\}$? Explain clearly. (3+3+2 marks)

a) $$[B \quad AB] = \begin{bmatrix} a v_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix} \Rightarrow \text{rank} = 2$$

$$A - BK = \begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a v_0/b \\ v_0/b \end{bmatrix} [k_1 \quad k_2]$$

$$= \begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{a v_0}{b} k_1 & \frac{a v_0}{b} k_2 \\ \frac{v_0}{b} k_1 & \frac{v_0}{b} k_2 \end{bmatrix} = \begin{bmatrix} -\frac{a v_0}{b} k_1 & v_0 - \frac{a v_0 k_2}{b} \\ -\frac{v_0}{b} k_1 & -\frac{v_0}{b} k_2 \end{bmatrix}$$

eigenvalues are roots of

$$\left(s+\frac{a v_0}{b}k_1\right)\left(s+\frac{v_0}{b}k_2\right)+\frac{v_0}{b}k_1\left(v_0-\frac{a v_0}{b}k_2\right)=0$$

$$\Rightarrow s^2+\left(\frac{a v_0}{b}k_1+\frac{v_0}{b}k_2\right)s+\frac{v_0^2}{b}k_1=0$$

$$\Rightarrow \quad k_1=b\frac{\omega_c^2}{v_0^2}, \quad \frac{a \cancel{v_0}}{\cancel{b}}\frac{\cancel{b}\,\omega_c^2}{v_0^{\cancel{2}}}+\frac{v_0}{b}k_2=2\zeta_c\omega_c$$

$$\Rightarrow k_2=\frac{b}{v_0}\left(2\zeta_c\omega_c-\frac{a\,\omega_c^2}{v_0}\right)$$

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b) $\begin{bmatrix} C \\ CA \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & v_0 \end{bmatrix}$ rank = 2

1

Observer $\dot{\hat{x}}=(A-LC)\hat{x}+Ly$

$$A-LC=\begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix}-\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix}-\begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}=\begin{bmatrix} -l_1 & v_0 \\ -l_2 & 0 \end{bmatrix}$$

eigenvalues are roots of $s^2+l_1 s+l_2 v_0=0$

$$\Rightarrow l_1=2\zeta_o\omega_o, \quad l_2=\frac{\omega_o^2}{v_0}$$

2

c) Would want these to be stable. Large eigenvalues need larger gain. Observer eigenvalues should be faster.

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