ELL 707
02.01.2020

Example 2:

$$\frac{dr}{dt} \rightarrow \dot{r} = r(1-r^2) \quad , r(0)$$

$$\frac{d\theta}{dt} \rightarrow \dot{\theta} = 1 \quad , \theta(0)$$

$$\Rightarrow \theta(t) = t + \theta(0)$$

$$\frac{dr}{dt} = r(1-r^2)$$

$$\Rightarrow \frac{dr}{r(1-r^2)} = dt$$

$$\Rightarrow dr\left[\frac{1}{r} + \frac{1}{2}\frac{1}{1-r} - \frac{1}{2}\frac{1}{1+r}\right] = dt$$

Integrate $\int_{r(0)}^{r(t)} \quad \int_0^t$

$$\Rightarrow \ln r\Big|_{r(0)}^{r(t)} - \frac{1}{2}\ln(1-r)\Big|_{r(0)}^{r(t)} - \frac{1}{2}\ln(1+r)\Big|_{r(0)}^{r(t)} = t$$

$$\Rightarrow 2\ln\frac{r(t)}{r(0)} - \ln\frac{1-r^2(t)}{1-r^2(0)} = 2t$$

$$\Rightarrow \ln \frac{r^2}{1-r^2}\cdot\underbrace{\frac{1-r^2(0)}{r^2_{(0)}}}_{\gamma_c} = 2t$$

$$\Rightarrow \quad \frac{r^2}{1-r^2} \cdot \frac{1}{c} = e^{2t}$$

$$\Rightarrow \quad \frac{r^2}{1-r^2} = c\, e^{2t}$$

$$\Rightarrow \quad \frac{1-r^2}{r^2} = \frac{1}{c} e^{-2t}$$

$$\Rightarrow \quad \frac{1}{r^2} - 1 = \frac{1}{c} e^{-2t}$$

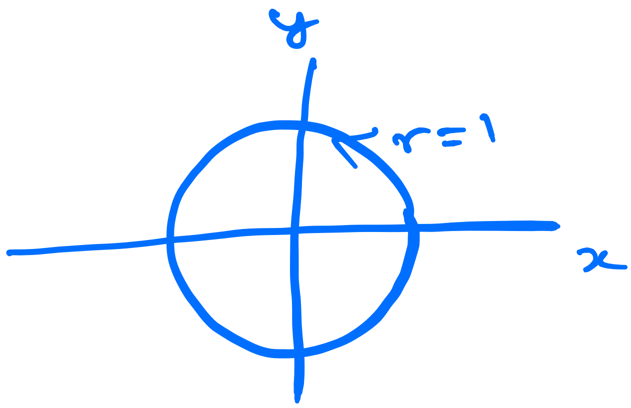
$$\Rightarrow \quad \frac{1}{r^2} = 1 + \frac{1}{c} e^{-2t}$$

$$\Rightarrow \quad r^2 = \frac{1}{1+\frac{1}{c}\cdot e^{-2t}}$$

$$\Rightarrow \quad r(t) = \sqrt{\frac{1}{1+\frac{1}{c} e^{-2t}}} \ , \quad c = \frac{r^2(0)}{1-r^2(0)}$$

$$\lim_{t\to\infty} r(t) = 1$$

$$\theta(t) = t + \theta(0)$$



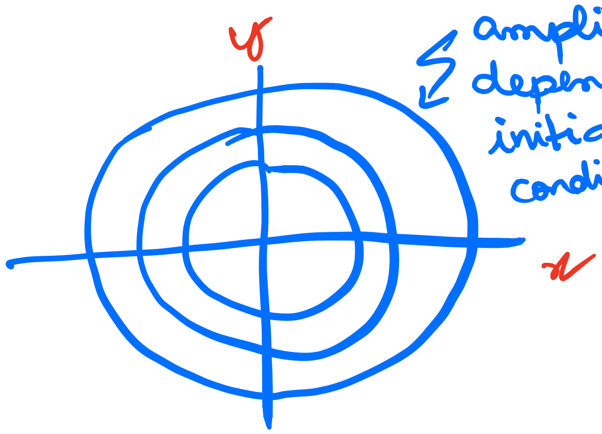
Contrast previous two examples

#1 $\dot{r} = 0$

$\dot{\theta} = 1$

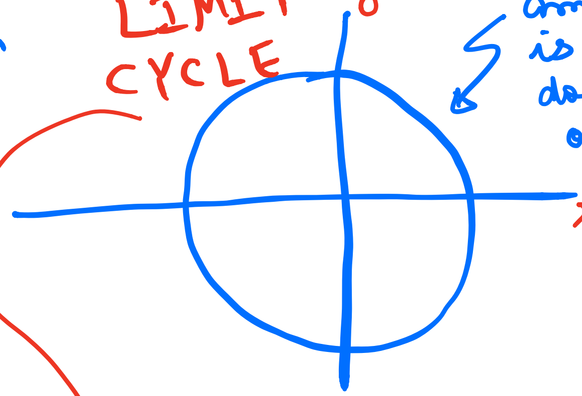
#2 $\dot{r} = r(1-r^2)$

$\dot{\theta} = 1$



amplitude depends on initial condition

LIMIT CYCLE



amplitude is fixed, doesn't depend on initial condition

$x(t) = r(0)\cos(t+\theta(0))$
$y(t) = r(0)\sin(t+\theta(0))$

$x(t) \to \cos(t+\theta(0))$
$y(t) \to \sin(t+\theta(0))$

$\rightarrow$ Oscillation is a limiting behaviour.

[Ref: S. Strogatz
Nonlinear Dynamical Systems & Chaos]

Can we get some of this information without explicitly solving the equations?

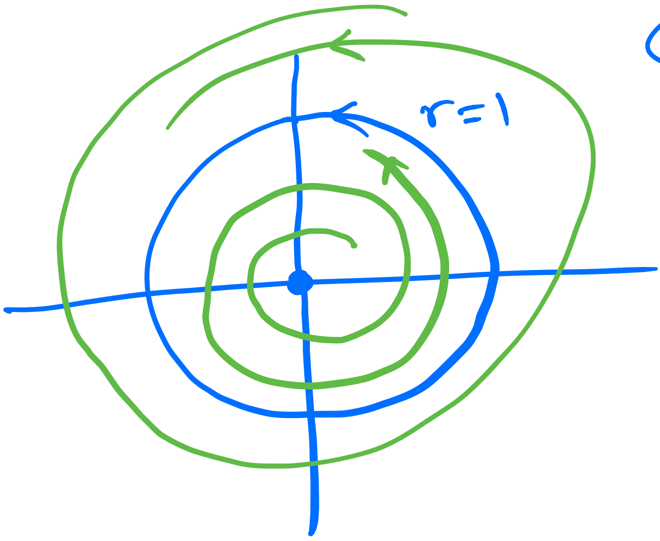
$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = 1$$

Suppose we look for solutions where $\dot{r} = 0$

idea: if $\dot{r} = 0$, $r = r(0)$ and it stays there

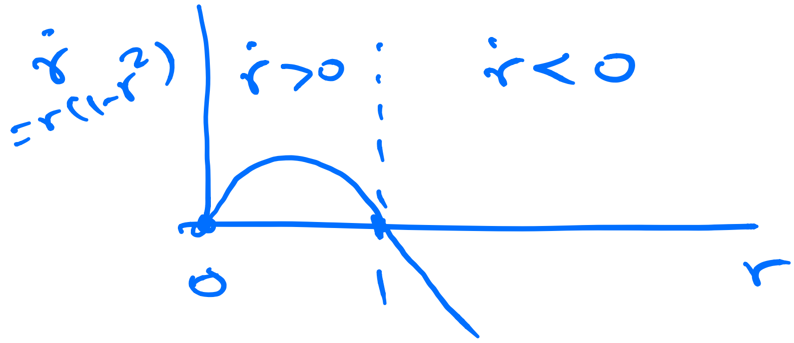
$\dot{r} = 0 \Rightarrow r(1-r^2) = 0$
$\Rightarrow r = 0$ or 1

[These are the <u>steady-state solutions</u> in the sense that if we start at them initially, we stay there]

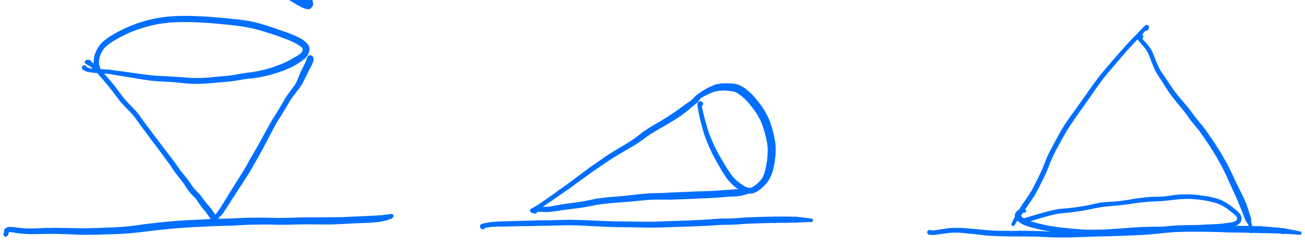


Q: Do solutions converge to this? (like we know from exact solution)

Check sign of $\dot{r}$ for $0<r<1$, $r>1$



Stability or Long-term behaviour



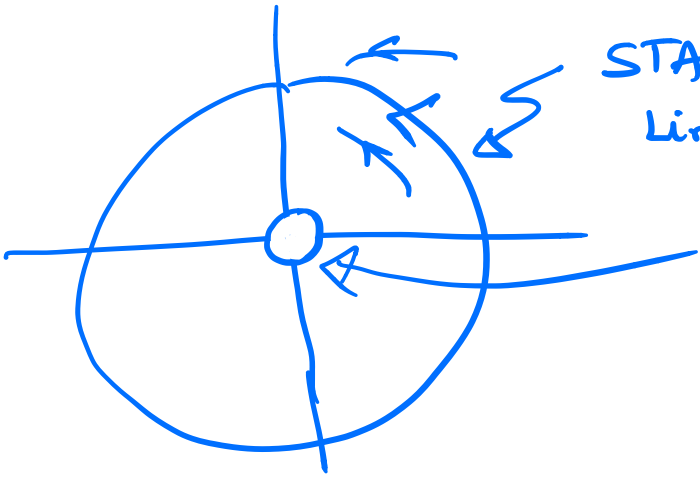
unstable

stable

"Lyapunov" Stability

~ If a solution is slightly perturbed, does it stay nearby or converge to the solution?

If yes, then stable in the sense of Lyapunov.



Example 3: Find fixed points and limit cycles for

$$\dot{r} = -r(1-r^2)$$

$$\dot{\theta} = 1$$

and analyse their stability.