ELL707

23.01.2020

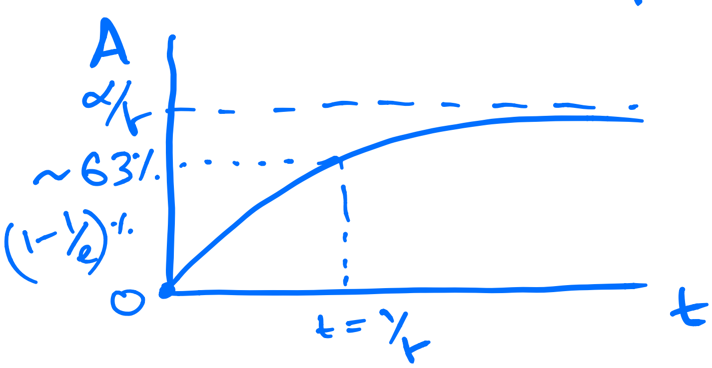
$$\frac{dA}{dt} = \alpha_0 + \alpha_1 \frac{K^n}{K^n + A^n} - rA \equiv A \xrightarrow{r} \phi$$

for comparison,

$$\frac{dA}{dt} = \alpha - rA \equiv \xrightarrow{\alpha} A \xrightarrow{r} \phi$$

Focus on ($\downarrow$) (check)

$$\Rightarrow A(t) = \frac{\alpha}{r}\left(1 - e^{-rt}\right), \quad A(0) = 0$$

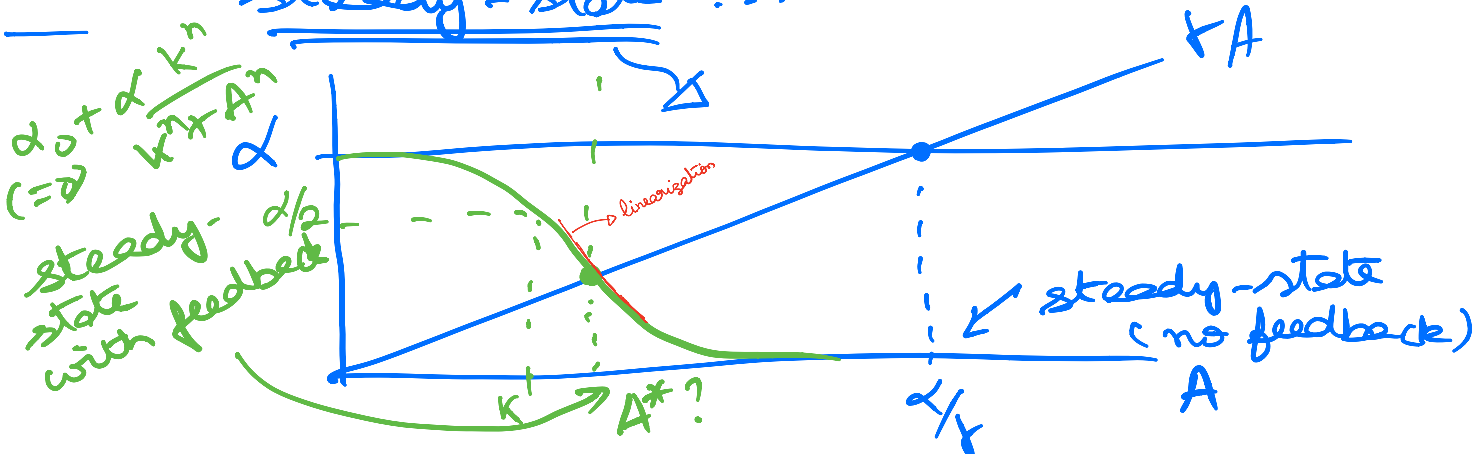


exponential response

time constant: $\frac{1}{r}$

Question: How does adding negative feedback, as in the above model, change the response time?

steady-state ...



What is A^* relative to α/γ ?

$$A^* > \alpha/\gamma \quad \text{or} \quad A^* < \alpha/\gamma$$

- $A^* < \frac{\alpha}{\gamma}$ always.
- If $K << \alpha/\gamma$, $A^* << \alpha/\gamma$
- If $K >> \alpha/\gamma$, $A^* \lesssim \alpha/\gamma$

<u>Response time comparison</u>

$(\alpha_0 = 0)$

$$\frac{dA}{dt} = \alpha \frac{K^n}{K^n + A^n} - \gamma A$$

How to do ?

→ Linearization approach

Linearize the nonlinear term about an operating.

Replace $A = A^* + \Delta A$ in ( )

LHS $\frac{dA}{dt} = \frac{d}{dt}(A^* + \Delta A) = \frac{d}{dt}\Delta A$

RHS $\frac{\alpha K^n}{K^n + A^n} - \gamma A = \frac{\alpha K^n}{K^n + (A^* + \Delta A)^n} - \gamma(A^* + \Delta A)$

$\cong$ Taylor Series of first term upto first order

$$\frac{\alpha K^n}{K^n + (A^*)^n} + \frac{d}{dA}\left(\frac{\alpha K^n}{K^n + A^n}\right)\Bigg|_{A=A^*} (\Delta A) \quad - rA^* - r\Delta A$$

$$= \underbrace{\frac{\alpha K^n}{K^n + (A^*)^n} - rA^*}_{=0} + \frac{-\alpha K^n . n (A^*)^{n-1}}{(K^n + (A^*)^n)^2} . \Delta A - r\Delta A$$

$\to = 0$ because A^* is a steady-state

$$= -\left(r + \frac{\alpha K^n . n (A^*)^{n-1}}{(K^n + (A^*)^n)^2}\right) \Delta A$$

linearization of the negative feedback model at $A = A^*$ is

$$\frac{d}{dt}\Delta A = -\left(r + (\text{sth +ve})\right)\Delta A$$

Repeat $A = \frac{\alpha}{r} + \Delta A$ for $\frac{dA}{dt} = \alpha - rA$

$$\frac{d}{dt}\Delta A = -r\Delta A$$

$\to$ Compare the time constants of these.

Which is faster?

As (sth +ve) > 0, negative feedback is locally faster than no feedback.

How much faster?

This depends on parameters.

If $K >> \frac{\alpha}{r} > A^*$

then sth twe $= \frac{\alpha K^n \cdot n (A^*)^{n-1}}{\left(K^n + (A^*)^n\right)^2}$

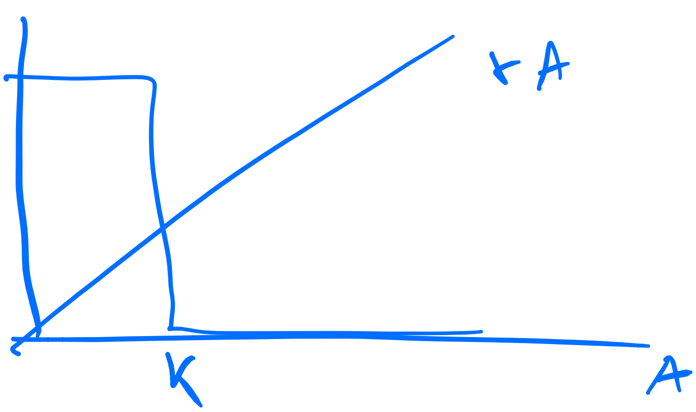
$$\approx \frac{\alpha \cancel{K^n} \cdot n (A^*)^{n-1}}{\cancel{K^n} \cdot K^n}$$

$$= \begin{cases} \frac{n\alpha}{K} \cdot \left(\frac{A^*}{K}\right)^{n-1} \sim 0, & n>1 \\ \frac{\alpha}{K} & \text{if } n=1 \end{cases}$$

$<< r$ in above limit

Small K. limit

$$\frac{dA}{dt} = \alpha \underbrace{\frac{K^n}{K^n + A^n}}_{\approx \alpha \cdot \frac{K^n}{A^n}} - rA$$



Say $K = A^*$

$$\text{sth tve} = \frac{\alpha\, n\, K^n (A^*)^{n-1}}{(K^n + (A^*)^n)^2}$$

$$\sim \frac{n \cdot \alpha \cdot K^n \; K^{n-1}}{(2K^n)^2}$$

$$= \frac{2^2 n \cdot \alpha}{K}$$

$$\frac{dA}{dt} = \alpha \frac{K^n}{K^n + A^n} - rA$$

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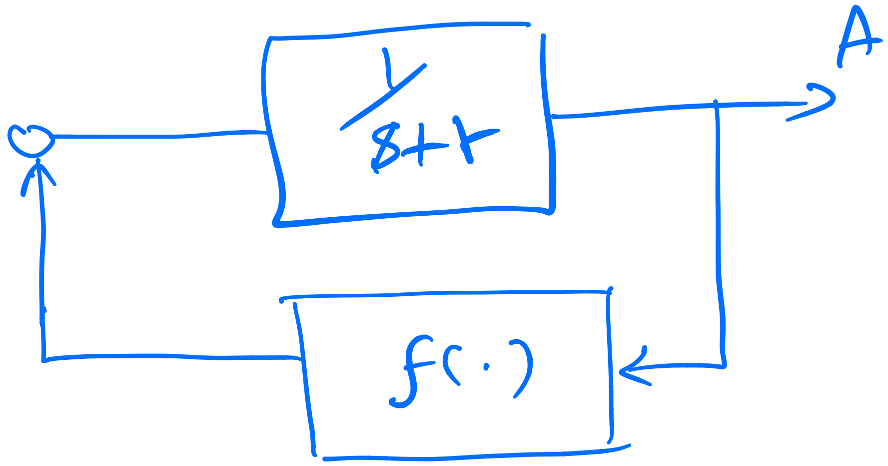
in ELL225 terms:



$$f(A) = \alpha \frac{K^n}{K^n + A^n}$$

$\equiv$ linearized around $A = A^*$