ELL707

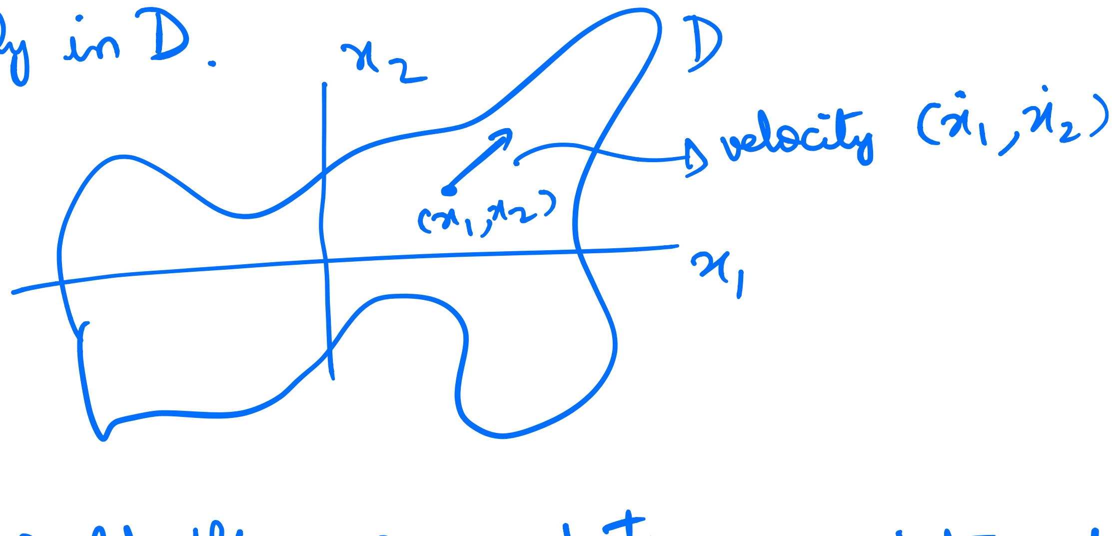
13.02.2020

Some theoretical results to prove/disprove limit cycle oscillations $\rightarrow$ Sec 3.3

1. Bendixson Criterion

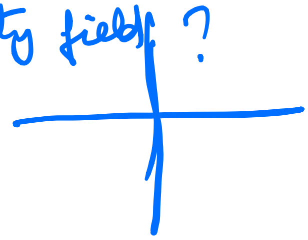
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2) \quad , \quad f \triangleq \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}_{2\times 1}$$

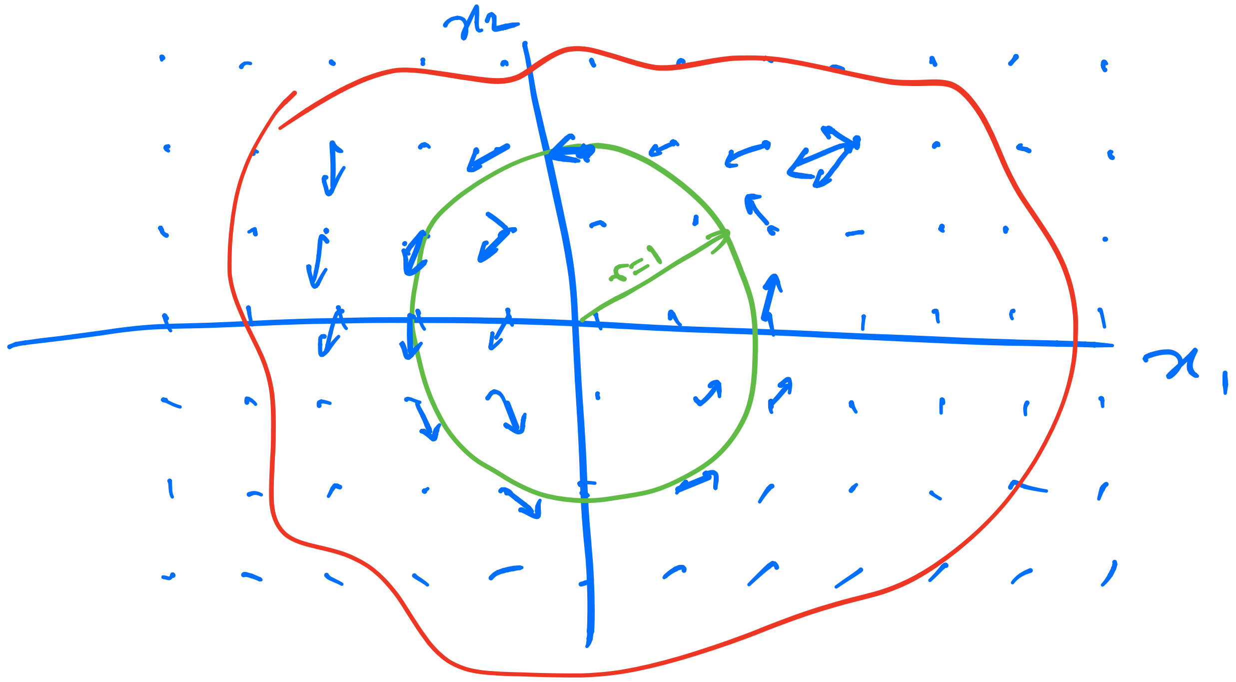
If $\nabla \cdot f = \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in a simply connected region $D \subset \mathbb{R}^2$, then there is no limit cycle trajectory entirely in D.



Why should this correspond to non-existence of a limit cycle? velocity field?

$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = 1$$





Problem: Convert from $(r, \theta) \rightarrow (x, y)$ and check the divergence

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Ex

$$\dot{x}_1 = \overbrace{u + K x_2 - x_1}^{f_1}$$
$$\dot{x}_2 = \underbrace{x_1 - x_2}_{f_2}$$

[N=2 in MTI, 4 (b)

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -1 - 1 = -2$$

$\Rightarrow$ no limitcycle in $\mathbb{R}^2$ for any K.

As you look at two dimensional models of biomolecular oscillators, check whether this criterion is applicable. $\hookrightarrow$ project paper.

Poincare - Bendixson Theorem

(Brief summary)

↳ applicable 2 dimensional

∵ Hilbert's 16th Problem

↳ Find a region M that is

a) closed + bounded

b) positively invariant

⇒ if there is no equilibrium point in M, then there is a limit cycle.

For higher dimensions results exist for a class of systems.

(Mallet-Paret & Smith)

$$\left.\begin{aligned}
\dot{x}_1 &= f_1(x_N, x_1) \\
\dot{x}_2 &= f_2(x_1, x_2) \\
\dot{x}_3 &= f_3(x_2, x_3) \\
&\vdots \\
\dot{x}_N &= f_N(x_{N-1}, x_N)
\end{aligned}\right\} \text{cyclic system}$$

Similar to Poincare Bendixson theorem statement

Ex: Cyclic system we started with

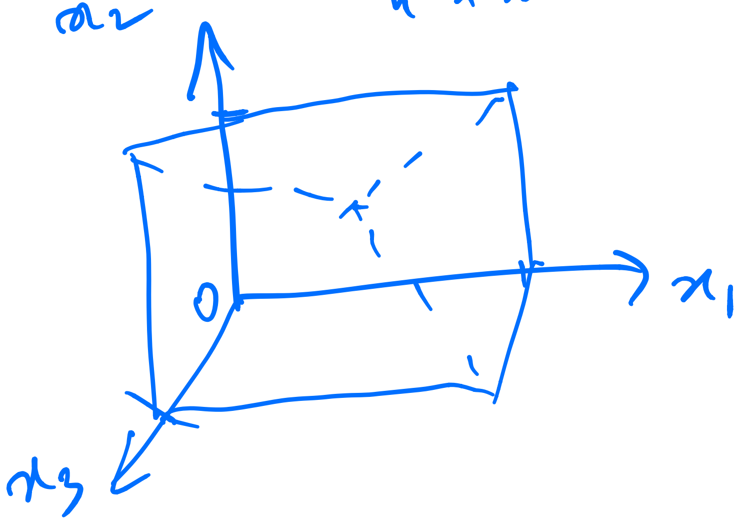
$$\dot{x}_1 = f(x_3) - r x_1$$

$$\dot{x}_2 = f(x_1) - r x_2$$

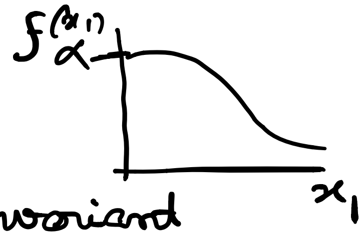
$$\dot{x}_3 = f(x_2) - r x_3$$

system in $\mathbb{R}^3$

$$f(x) = \alpha_0 + \frac{\alpha K^n}{K^n + x^n}$$



<u>n=1</u>



$$\dot{x}_1 = f(x_1) - r x_1$$

Claim : $[0, \frac{\alpha}{r}]$ is positively invariant

at $x_1 = 0$, $\dot{x}_1 = \alpha > 0 \Rightarrow x_1 \uparrow$

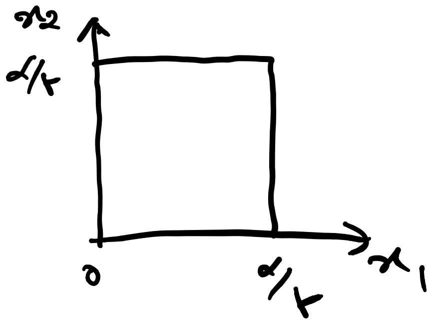
at $x_1 = \frac{\alpha}{r}$, $\dot{x}_1 = f(x_1) - \alpha < 0 \Rightarrow x_1 \downarrow$ $\square$

<u>n=2</u>

$$\dot{x}_1 = f(x_2) - r(x_1)$$

$$\dot{x}_2 = f(x_1) - r x_2$$

Claim : $[0, \frac{\alpha}{r}] \times [0, \frac{\alpha}{r}]$ is positively invariant.



<u>$x_2 = 0$</u> $\dot{x}_1 = \alpha - r x_1 \geq 0$ for $x_1 \in [0, \frac{\alpha}{r}]$

$\dot{x}_2 = f(x_1) > 0$

<u>$x_1 = 0$</u> $\dot{x}_1 = f(x_2) > 0$

$\dot{x}_2 = \alpha - r x_2 \geq 0$ for $x_2 \in [0, \frac{\alpha}{r}]$

$x_1 = \alpha/r$, $\dot{x}_1 = f(x_2) - \alpha < 0$

$\dot{x}_2 = f(x_1) - r x_2 \gtrless 0$

$x_2 = \alpha/r$, $\dot{x}_1 = f(x_2) - r x_1 \gtrless 0$

$\dot{x}_2 = f(x_1) - \alpha < 0$

May work for n=3 also?