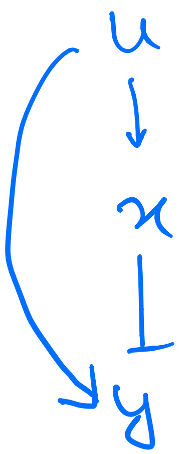
ELL 707

27.02.2020

Example 1:



$$\frac{dx}{dt} = \alpha_x u - r_x x$$

$$\frac{dy}{dt} = \alpha_y u \frac{K}{x} - r_y y$$

exact solution?

- steady-state

$$x^* = \frac{\alpha_x}{r_x} u$$

$$y^* = \frac{\alpha_y}{\alpha_x} \cdot \frac{r_x}{r_y} \cdot K$$

no u here

y^* is independent of exact value of u ($\neq 0$)

- stability

Linearize around (x^*, y^*)

$$x = x^* + \Delta x, \quad y = y^* + \Delta y$$

$$\Rightarrow \quad \frac{d\Delta x}{dt} = \alpha_x u - r_x (x^* + \Delta x)$$

$$\& \quad \frac{d\Delta y}{dt} = \alpha_y u \cdot \frac{K}{x^* + \Delta x} - r_y (y^* + \Delta y)$$

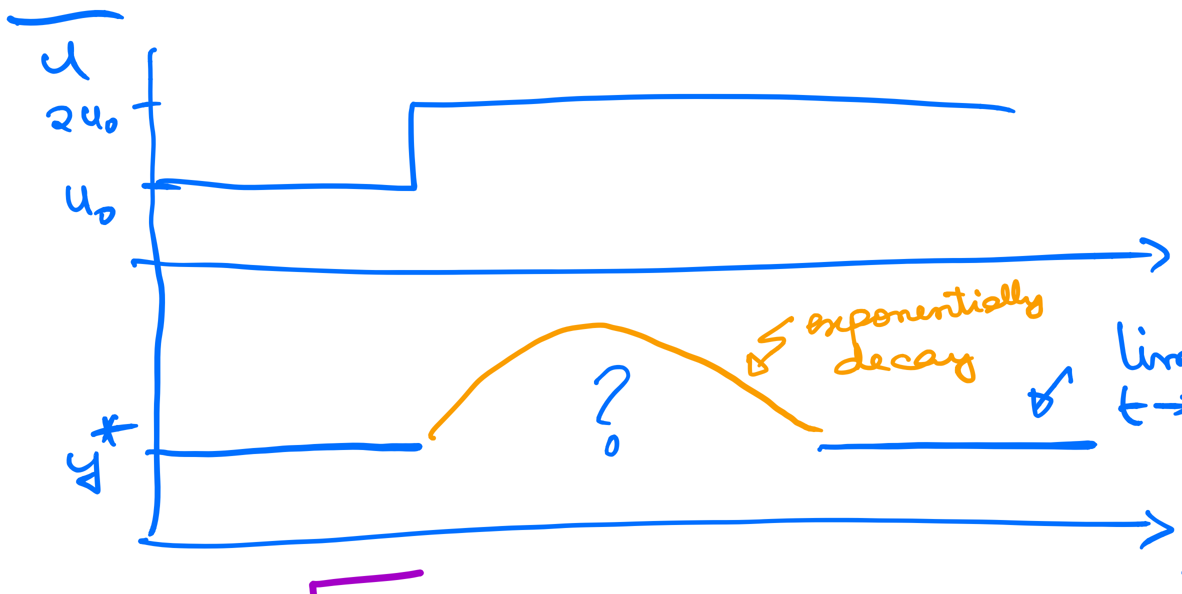
$$\Rightarrow \frac{d\Delta x}{dt} = -r_x \Delta x$$

$$\& \quad \frac{d\Delta y}{dt} = \alpha_y u \frac{K}{x^*} - \alpha_y u \frac{K}{(x^*)^2} \Delta x - r_y y^* - r_y \Delta y$$

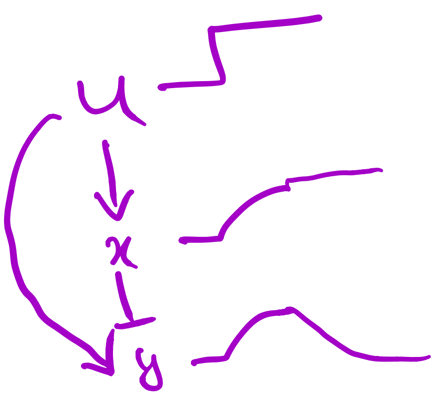
$$= -\alpha_y u \frac{K}{(x^*)^2} \Delta x - r_y \Delta y$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \underbrace{\begin{bmatrix} -r_x & 0 \\ -\alpha_y u \frac{K}{(x^*)^2} & -r_y \end{bmatrix}}_{\downarrow} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

Stable because eig( ) $= -r_x, -r_y < 0$



$\lim_{t \to \infty} y(t) = y_0$ ✓

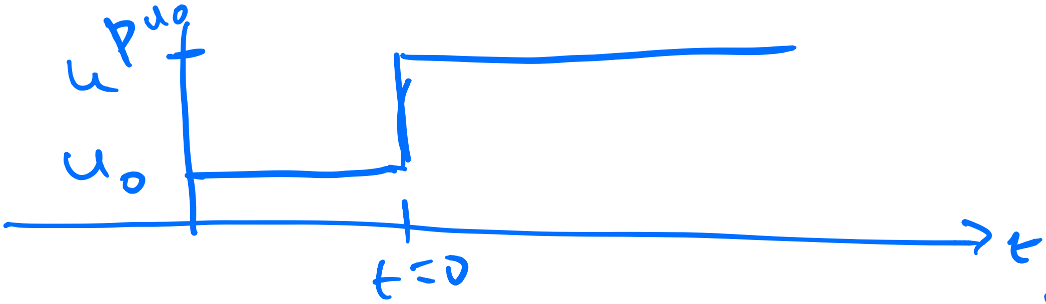


? because $\frac{u}{x}$ can be anything ?

For more rigorous explanation, lets try to solve?

$$\frac{dx}{dt} = \alpha_x u - r_x x$$

$$\frac{dy}{dt} = \alpha_y u \frac{K}{x} - r_y y$$



$$x(0) = \frac{\alpha_x u_0}{r_x}, \quad y(0) = \frac{\alpha_y}{\alpha_x} \cdot \frac{r_x}{r_y} \cdot K$$

$$x(t) = x(0) e^{-r_x t} + \frac{\alpha_x}{r_x} \cdot p u_0 (1 - e^{-r_x t})$$

$$= \frac{\alpha_x}{r_x} \cdot u_0 e^{-r_x t} + \frac{\alpha_x}{r_x} \cdot p u_0 \cdot (1 - e^{-r_x t})$$

$$\frac{dy}{dt} = \alpha_y p u_0 \cdot \frac{K}{x(t)} - r_y y$$

$$\frac{dy}{dt} = \frac{\alpha_y \cdot p u_0 \cdot K}{\frac{\alpha_x}{r_x} \cdot u_0 e^{-r_x t} + \frac{\alpha_x}{r_x} \cdot p u_0 \cdot (1 - e^{-r_x t})} - r_y y$$

$\frac{dy}{dt} + r_y y$ & move $r_y y$ to other side

$$\frac{d\, e^{r_y t} \cdot y}{dt} = \frac{\alpha_y\, p u_0 K \quad e^{r_y t}}{\frac{\alpha_x}{r_x} u_0 \left[p + (1-p) e^{-r_x t} \right]}$$

$$d\, e^{r_y t} y = \underbrace{\left[\frac{\alpha_y}{\alpha_x} \cdot r_x \cdot p \cdot K \right]}_{c} \cdot \frac{e^{r_y t} \quad dt}{p + (1-p)\, e^{-r_x t}}$$

other way: homogeneous soln. + particular soln.

Ritwik will try

$$d\, e^{r_y t} y = c \; \frac{e^{(r_y + r_x) t} \quad dt}{1 - p + p\, e^{r_x t}}$$

For $r_y = r_x = r$

$$\Rightarrow d\, e^{rt} y = c \; \frac{e^{rt} \; e^{rt} \; dt}{1 - p + p e^{rt}}$$

$$z = 1 - p + p e^{rt} \qquad \Rightarrow dz = p e^{rt} \cdot r\, dt$$

$$\Rightarrow e^{rt} dt = dz \cdot \frac{1}{pr}$$

$$\& \quad e^{rt} = \frac{z - 1 + p}{p}$$

$$\Rightarrow d\, e^{rt} y = c \cdot \left(\frac{z - 1 + p}{p} \right) \cdot \frac{1}{z} \cdot \frac{dz}{pr}$$

$$\Rightarrow d e^{rt} y = \frac{c}{p^2 r} \left[\frac{z - 1 + p}{z} \right] dz \longrightarrow \left[1 + \frac{p-1}{z} \right]$$

Integrate $t : 0 \longrightarrow t$

$y : y(0) \longrightarrow y$

$= \frac{\alpha_y}{\alpha_x} \frac{r_x}{r_y} . k$

$z : 1 \longrightarrow 1 - p + p e^{rt}$

$$e^{rt} y - y(0) = \frac{c}{p^2 r} \left[z \Big|_1^{1-p+pe^{rt}} + (p-1) \ln z \Big|_1^{1-p+pe^{rt}} \right]$$

$$\Rightarrow e^{rt} y - y(0) = \frac{c}{p^2 r} \left[p(e^{rt} - 1) + (p-1) \ln(1-p+pe^{rt}) \right]$$

$$\Rightarrow e^{rt} y = y(0) + \frac{c}{pr}(e^{rt} - 1) + \frac{c}{p^2 r}(p-1) \ln(1-p+pe^{rt})$$

$$\Rightarrow y(t) = y(0) e^{-rt} + \frac{c}{pr}(1 - e^{-rt}) + \frac{c}{r} \frac{p-1}{p^2} e^{-rt} \ln(1-p+pe^{rt})$$

$$\Rightarrow y(t) = \frac{\alpha_y}{\alpha_x} . k e^{-rt} + \frac{\alpha_y}{\alpha_x} . k (1 - e^{-rt}) + \frac{\alpha_y}{\alpha_x} . k . \frac{p-1}{p} e^{-rt} \ln(1-p+pe^{rt})$$

$\underbrace{\frac{\alpha_y}{\alpha_x} . r_x . p . K}_{c}$

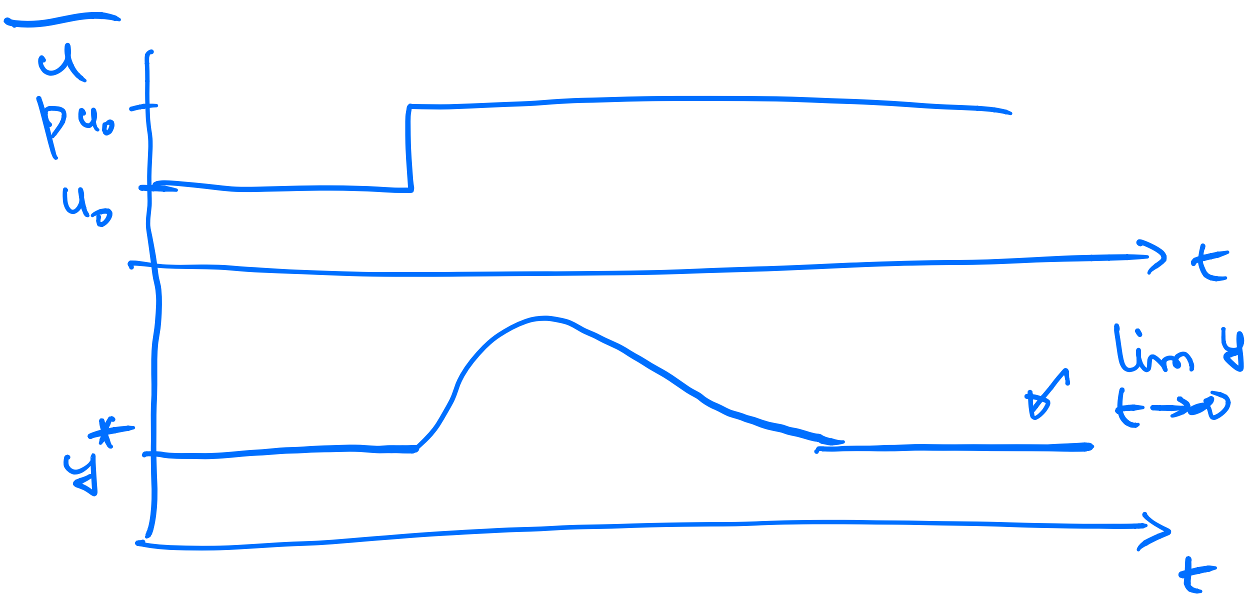
$$\Rightarrow y(t) = \underbrace{\frac{\alpha_y}{\alpha_x} \cdot K}_{y^*} + \underbrace{\frac{\alpha_y}{\alpha_x} \cdot K \cdot \frac{p-1}{p}\, e^{-rt} \ln(1-p+pe^{rt})}_{\text{transient response}}$$

as $t \to \infty$, dominated by exponential decay

for $t \sim 0^+$, ?

e^{-rt} : decreasing

$\ln(1+p(e^{rt}-1))$: increasing



$\lim_{t\to\infty} y(t) = y_0$ ✓