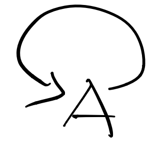
ELL 707 MINOR TEST I 17 marks, 1 hour

1. State in a sentence, what you plan to investigate in the course project. $\boxed{\text{1 mark}}$

2. Consider the following model of a circuit with positive feedback,



$$\frac{dA}{dt} = f(A) - \gamma A, \quad \text{— } ①$$

where A (>0) is the protein concentration, $f(A)$ is a non-decreasing function of A, & $\gamma > 0$ is the degradation constant.

a) For $f(A) = \alpha \cdot \dfrac{A^n}{K^n + A^n}$, where $\alpha (>0)$ is the maximum production rate, $K (>0)$ is the dissociation constant, and n (integer ≥ 2) is the Hill co-efficient, find the number of stable steady-states of ①. (illustrate graphically)

b) What $f(A)$ could be chosen if it is desired to have exactly three distinct stable steady-states and some (≥ 0) unstable steady-states. $\boxed{\text{5 marks} = 3 + 2}$

3. Consider the model $\dot{r} = r(2-r^2)(3-r^2)(5-r^2)$,

$\dot{\theta} = -1.$ — (2)

a) Identify limit cycles and analyse their stability.

b) For (2), the Poincaré-Bendixson Theorem can be used to show that { if ← statement

i) R is a closed, bounded subset of X-Y plane,

ii) R does not contain any steady-state (a point where $\dot{x} = \dot{y} = 0$).

iii) solutions to (2) that are in R at $t=0$ remain in R for all $t \geq 0$,

then R contains a limit cycle. }

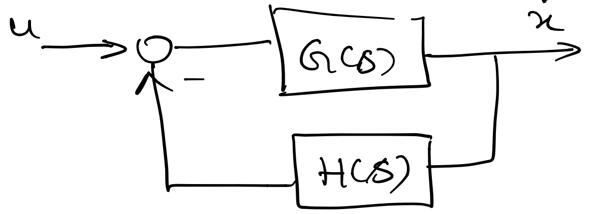
Prove the existence of limit cycle solution(s) for (2) by identifying suitable R.

$\boxed{\text{5 marks} = 2+3}$

4. The goal of this problem is to investigate the role of delayed negative feedback in generating oscillations.

a) Consider the differential equation

$\ddot{x} + \omega_0^2 x = \underline{u}.$ (3) Show that it can be represented in Laplace domain by



where $G(s) = \frac{1}{s}$ and $H(s) = \frac{\omega_0^2}{s}$.

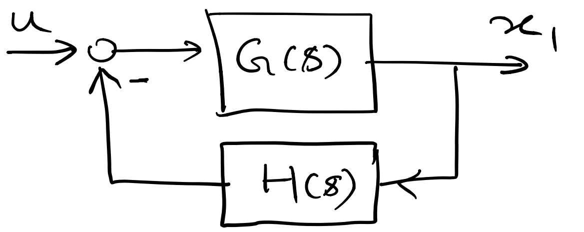
{Note that the denominator polynomial of $\frac{G(s)}{1+G(s)H(s)}$ has purely imaginary roots, the $D(s) = s^2 + \omega_0^2$, $D(s) = 0 \Rightarrow s = \pm j\omega_0$ locations of which correspond to the oscillation frequency of solutions of (3).}

b) Consider the cyclic system of differential equations,

$$\dot{x}_1 = u - K x_N - x_1, \quad K \in \mathbb{R} \quad (4)$$

$$\dot{x}_i = x_{i-1} - x_i, \quad i = 2, 3, \ldots N$$

i) Show that (4) can be represented as



where $G(s) = \frac{1}{s+1}$.

Find $H(s)$.

(depends on N and K.)

ii) Show that the denominator polynomial of $\frac{G(s)}{1+G(s)H(s)}$ is $D(s) = (s+1)^N + K$.

iii) Show that $D(s) = 0$ has no purely imaginary pair of roots ($\pm j\omega, \omega \neq 0$) if $K = 0$.

iv) Show that $D(s) = 0$ has no purely imaginary pair of roots ($\pm j\omega$, $\omega \neq 0$) for $N = 2$ and any $K \in \mathbb{R}$

v) Show that $D(s) = 0$ has a pair of purely imaginary roots ($\pm j\omega, \omega \neq 0$) for $N = 3$ and some particular K that has to be positive.

$$\boxed{6 \text{ marks} = 1 + \underbrace{5}_{1+1+1+1+1}}$$