State Space Model Equations

$\rightarrow$ n states, m inputs

- $\underset{n\times 1}{\dot{x}} = \underset{?}{A}\, \underset{n\times 1}{x} + Bu \qquad x \in \mathbb{R}^{n\times 1},\ u \in \mathbb{R}^{m\times 1},\ y \in \mathbb{R}^{p\times 1} \rightsquigarrow$ p outputs

  $y = Cx + Du \qquad A \in \mathbb{R}^{\underline{n\times n}},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$

  Example: $x_1 = \phi,\ x_2 = \dot{\phi},\ x_3 = \delta,\ x_4 = \dot{\delta},\ u = f$

  $$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ * & * & * & * \\ 0 & 0 & 0 & 1 \\ * & * & * & * \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & * \\ 0 & 0 \\ * & * \end{bmatrix}_{4\times 2}\begin{bmatrix} T_\phi \\ T_\delta \end{bmatrix}$$

  Check!

- Matrix / Linear Algebra

  determinants, inverse $inv(A) = \frac{adj(A)}{det(A)}$, vector space, eigenvalues & eigenvectors, matrix exponential, Cayley-Hamilton Theorem, rank, nullspace, rangespace.
  (singular value decomposition)

- "The matrix is everywhere".

Have Equation, Will Solve?

- How can we solve?

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Can we solve?

- Why should we solve?

→ Transient response / DYNAMICS {how fast does a standing bicycle fall?}

→ Long-term behaviour / STABILITY

{If a system is perturb-ed, does it return to original state or does it go away?}

Time

- We are looking for solutions as functions of time.
- $t \in \mathbb{R}$. Typically, initial time, $t_0 = 0$.
- What is time?

Existence & Uniqueness

- Theorems such as these...

- Local

Introduction to Applied Nonlinear Dynamical Systems and Chaos, Stephen Wiggins

Consider the vector field $\dot{x} = f(x,t)$ where $f(x,t)$ is C^r, $r \geq 1$, on some open set $U \in \mathbb{R}^n \times \mathbb{R}^1$.

<u>Theorem 7.1.1</u> Let $(x_0, t_0) \in U$. Then **there exists a solution** of $\dot{x} = f(x,t)$ through the point x_0 at $t = t_0$, denoted $x(t, t_0, x_0)$ with $x(t_0, t_0, x_0) = x_0$, for $|t - t_0|$ sufficiently small. This **solution is unique** in the sense that any other solution of $\dot{x} = f(x,t)$ through x_0 at $t = t_0$ must be the same as $x(t, t_0, x_0)$ on their common interval of existence. Moreover, $x(t, t_0, x_0)$ is a C^r function of t, t_0, and x_0.

- Global

Let $C \subset U \subset \mathbb{R}^n \times \mathbb{R}^1$ be a compact set containing (x_0, t_0).

<u>Theorem 7.2.1</u> The solution $x(t, t_0, x_0)$ can be uniquely **extended** backward and forward in t up to the boundary of C.

Matrix Exponential

- Solution of $\dot{x} = Ax + Bu$ is $x(t) = \underbrace{\underset{n\times n}{e^{At}}\,\underset{n\times 1}{x(0)}}_{n\times 1}$,

  $y = Cx + Du$

verify this statement

with $e^{At} \underset{\text{def}}{=} I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots + \frac{t^k}{k!}A^k$.

$\dot{x} = ax \Rightarrow \dot{x} - ax = 0 \Rightarrow \frac{d}{dt} x e^{-at} = 0 \;\dots\; x(t) = x(0)e^{at}$ generalised?

- Why? $\boxed{\frac{d}{dt}(e^{At}x(0)) = A(e^{At}x(0)) \Rightarrow \text{solves the equation.}}$ + uniqueness.

- Properties

  → $e^0 = I$ → $\frac{d}{dt}e^{At} = Ae^{At} (= e^{At}A)$. → $(e^{At})^{-1} = e^{-At}$

  → If X and Y commute ($XY = YX$), then $e^{X+Y} = e^X \cdot e^Y$.

  ↳ where is this needed?

  scalars: $e^{x+y} = e^x \cdot e^y$
  not always true for matrices!

Diagonalize $\rightarrow \mathcal{L}^{-1}((sI-A)^{-1})$

- Easy to compute e^{At} when A is diagonal: $e^{\overbrace{\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}}^{}_{A} t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}$

$A^2 = \begin{bmatrix}\lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2\end{bmatrix} \rightarrow e^{At} = \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} + t\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix}\lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2\end{bmatrix} + \dots$

e^{Dt} is known as D is diagonal

- If A is similar to a diagonal matrix D, then $\boxed{e^{At} = V e^{Dt} V^{-1}}$.
  (there exist V such that $\boxed{V^{-1} A V = D}$, A is diagonalizable)

  why? $e^{At} = \sum \frac{t^k}{k!} A^k$, $A^k = A \dots A = VDV^{-1} \cdot VDV^{-1} \dots VDV^{-1} = VD^kV^{-1}$

V is a co-ordinate transformation,
$z = V^{-1}x \Rightarrow \dot{z} = V^{-1}\dot{x} = V^{-1}Ax = V^{-1}AVz = Dz$

- A is diagonalizable if A has n distinct eigenvalues.

$\underbrace{Av_1 = \lambda_1 v_1,\ Av_2 = \lambda_2 v_2, \dots Av_n = \lambda_n v_n}$, $v_i \neq 0$, λ_i's distinct.
$i = 1, \dots, n$

$A[v_1\ v_2\ \dots\ v_n] = [v_1\ v_2\ \dots\ v_n] \begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix}$

Construction of V — $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix} \dots$ … any vector

Algorithm

$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = 1 \cdot \begin{bmatrix}3\\2\end{bmatrix}$ D eigenvalues: $1, -1$

- Is V invertible?
- eigenvalues / eigenvectors of $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$?

18.08.2022

Question

$A_{n\times n}$ $\quad A v = \lambda v, \; v \neq 0$

eigenvector $\quad$ eigenvalue

Solve $\underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial}} = 0$ for λ to get eigenvalues

- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

Eigenvalues? $\quad (\lambda-1)^2 = 0 \Rightarrow \lambda = 1, 1$

Eigenvectors? $\quad \text{nullspace}(A-\lambda I) = \{ v \in \mathbb{R}^{2\times 1} : \underset{2\times 2}{(A-\lambda I)} \underset{2\times 1}{v} = \underset{2\times 1}{0} \}$

$\lambda = 1$ $\quad A - \lambda I = \begin{bmatrix} 1-1 & 0 \\ 0 & 1-1 \end{bmatrix} = 0_{2\times 2}$

eigenvectors $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \ldots$

Matrix Exponential?

$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \ldots$

Set $A = I$, $\rightarrow \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}$

Construct Basis of eigenvectors $\{v_1, v_2\}$ $\leftarrow$ Part of the diagonalization algorithm

- $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

$J^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad J^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Eigenvalues? $\rightarrow 1, 1$

Eigenvectors? $\rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Matrix Exponential? $e^{Jt} = I + Jt + \frac{t^2}{2!}J^2 + \frac{t^3}{3!}J^3 + \ldots$

$$= \begin{bmatrix} 1 + t + \frac{t^2}{2!} + \ldots & t + \frac{t^2}{2!}\cdot 2 + \frac{t^3}{3!}\cdot 3 + \ldots \\ 0 & 1 + t + \frac{t^2}{2!} + \ldots \end{bmatrix}$$

e^t

$\rightarrow t\left[1 + t + \frac{t^2}{2!} + \ldots\right] = te^t$

e^t

e^{At} when $A \in \mathbb{R}^{2\times 2}$

- 2 distinct eigenvalues, $\lambda_1 \neq \lambda_2$

$\Rightarrow A v_1 = \lambda_1 v_1$, $A v_2 = \lambda_2 v_2$ & v_1, v_2 are linearly independent (V^{-1} should exist)

Assume linearly dependent
$\Rightarrow c_1 v_1 + c_2 v_2 = 0$, $c_1, c_2 \neq 0$ ①
$A \times$①
$\Rightarrow c_1 A v_1 + c_2 A v_2 = 0$
$\Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0$ ②
$\lambda_1 \times$① $-$ ② $\Rightarrow c_2(\lambda_1 - \lambda_2) v_2 = 0$

$$\Rightarrow A \underbrace{[v_1 \;\; v_2]}_{V} = \underbrace{[v_1 \;\; v_2]}_{V} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{D}$$

$$\Rightarrow AV = VD$$

$$\Rightarrow A = VDV^{-1}, \quad e^{At} = V e^{Dt} V^{-1}, \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

- 1 eigenvalue $\lambda_1 = \lambda_2 = \lambda$ (say), but two linearly independent eigenvectors v_1, v_2.

same as above

$$A \underbrace{[v_1 \;\; v_2]}_{V} = \underbrace{[v_1 \;\; v_2]}_{V} \underbrace{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{D}$$

$$\Rightarrow AV = VD$$

$$\Rightarrow A = VDV^{-1}, \quad e^{At} = V e^{Dt} V^{-1}, \quad e^{Dt} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}$$

- 1 eigenvalue and one eigenvector

$\lambda_1 = \lambda_2 = \lambda$ (say) $\qquad v_1 \Rightarrow A v_1 = \lambda v_1$

Construct Basis $\{v_1, v_2\}$, v_2 is LI from v_1

$$A v_2 = c_1 v_1 + c_2 v_2$$

$$A \underbrace{[v_1 \quad v_2]}_{V} = [A v_1 \quad A v_2] = [\lambda v_1 \quad c_1 v_1 + c_2 v_2]$$

$$= [v_1 \quad v_2] \begin{bmatrix} \lambda & c_1 \\ 0 & c_2 \end{bmatrix}_{2 \times 2}$$

<u>Goal</u> is to choose "simple enough" c_1, c_2.

If we pick $\quad$ does such a v_2 exist?

$$v_2 \in \text{nullspace}\{(A - \lambda I)^2\} \setminus \text{nullspace}\{(A - \lambda I)\}$$

generalised eigenvector $\quad (A - \lambda I)^2 v_2 = 0$ but $(A - \lambda I) v_2 \neq 0$

$$(A - \lambda I) \underbrace{(A - \lambda I) v_2}_{} = 0$$

Choose v_2 such that $(A - \lambda I)\, v_2 = v_1$

$$\Rightarrow A v_2 = v_1 + \lambda v_2$$

$c_1 = 1$ $\qquad$ $c_2 = \lambda$

$$\underbrace{A [v_1 \quad v_2]}_{V} = [A v_1 \quad A v_2] = [\lambda v_1 \quad c_1 v_1 + c_2 v_2]$$

$$= [v_1 \quad v_2] \begin{bmatrix} \lambda & c_1 = 1 \\ 0 & c_2 = \lambda \end{bmatrix}_{2 \times 2}$$

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad e^{Jt} = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

Jordan Block.

Quiz 1 18.08.2022

For the system $\ddot{\theta} - \theta = 0$,

(a) Obtain the state-space equations $\dot{x} = Ax$ by setting $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1 = \theta$, $x_2 = \dot{\theta}$.

(b) Find the eigenvalues of A.

(c) Find the eigenvectors of A.

(d) Find e^{At}.

22.08.2022

Question

- $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$ • $J_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$ • $J_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

Eigenvalues? Eigenvectors? Matrix Exponential?

Calculate these for all three matrices.

Jordan Canonical Form

C. Moler + C. Van Loan

- "Nineteen Dubious Ways to Compute the Exponential of a Matrix"

→ Series Definition → Cayley Hamilton Theorem → Inverse Laplace Transform

→ Diagonalization → Jordan Canonical Form → Schur Decomposi- -tion

• Generalized Eigenvectors

- "clever" expansion of basis of eigenvectors
- may have to search for these if there is a repeated eigenvalue λ_i
- If λ_i is repeated $r (\leq n)$ times and there are only $k (< r)$ linearly independent eigenvectors, then we have to find $r-k$ generalised eigenvectors that satisfy

$$\left.\begin{aligned} (A-\lambda_i I)^p v_{i_j} &\neq 0 \\ \text{but } (A-\lambda_i I)^q v_{i_j} &= 0 \end{aligned}\right\}$$

nullspace$\{(A-\lambda_i I)\}$ ← eigenvector

nullspace$\{(A-\lambda_i I)^2\}$ ← generalized eigenvectors

⋮

nullspace$\{(A-\lambda_i I)^r\}$

$j = 1, 2, \ldots r-k$

$p = 1, 2, \ldots j, \; q = j+1$

• Jordan Decomposition

$$A = VJV^{-1}, \quad J = \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_r \end{bmatrix}_{n\times n}$$

a Jordan block

$r \leq n$

check

$$\Rightarrow e^{At} = V e^{Jt} V^{-1}, \quad e^{Jt} = \begin{bmatrix} e^{\Lambda_1 t} & & & \\ & e^{\Lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\Lambda_r t} \end{bmatrix}_{n\times n}$$

$1\times1 \quad \Lambda_i = \lambda_i, \; e^{\Lambda_i t} = e^{\lambda_i t}$

λ_i repeated twice, but only one eigenvector

$2\times2 \quad \Lambda_i = \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}, \; e^{\Lambda_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} \\ 0 & e^{\lambda_i t} \end{bmatrix}$

$3\times3 \quad \Lambda_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}, \; e^{\Lambda_i t} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

2x2 $\Lambda_i = \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}, \quad e^{\Lambda_i t} = \begin{bmatrix} e^{\lambda_i t} & t e^{\lambda_i t} \\ 0 & e^{\lambda_i t} \end{bmatrix}$

$$e^{\Lambda_i t} = I + \Lambda_i t + \frac{t^2}{2!}\Lambda_i^2 + \cdots$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Lambda_i = \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}, \quad \Lambda_i^2 = \begin{bmatrix} \lambda_i^2 & 2\lambda_i \\ 0 & \lambda_i^2 \end{bmatrix}$$

$$\Lambda_i^3 = \begin{bmatrix} \lambda_i^3 & 3\lambda_i^2 \\ 0 & \lambda_i^3 \end{bmatrix}, \quad \Lambda_i^4 = \begin{bmatrix} \lambda_i^4 & 4\lambda_i^3 \\ 0 & \lambda_i^4 \end{bmatrix} \quad \cdots$$

$$= \begin{bmatrix} 1 + t\lambda_i + \frac{t^2}{2!}\lambda_i^2 + \cdots & 0 + t + \frac{t^2}{\cancel{2!}}2\lambda_i + \frac{t^3}{2!\,\cancel{3!}}\cancel{3}\lambda_i^2 + \cdots \\ 0 & \end{bmatrix}$$

$$t\left(1 + t\lambda_i + \frac{t^2}{2!}\lambda_i^2 + \cdots\right)$$

$$= \begin{bmatrix} e^{\lambda_i t} & t e^{\lambda_i t} \\ 0 & e^{\lambda_i t} \end{bmatrix}$$

polynomial

exponential

3x3 $\Lambda_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}, \quad e^{\Lambda_i t} = \begin{bmatrix} e^{\lambda_i t} & t e^{\lambda_i t} & \frac{t^2}{2!} e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & t e^{\lambda_i t} \\ 0 & 0 & e^{\lambda_i t} \end{bmatrix}$ to be checked

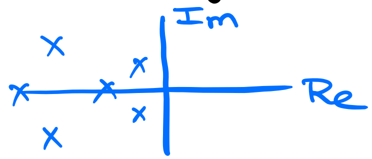
↑ complete solution

Stability → long term behaviour, $t \to \infty$, $x(t)$?

- Solution $x(t) = e^{At}x(0)$.

  Dynamics already computed: combinations of $\boxed{t^k e^{\lambda_i t}}$.

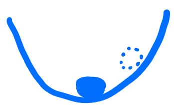
  Stability? If $Re\{\lambda_i\} < 0 \ \forall i$, then $\lim\limits_{t\to\infty} x(t) = 0$



because the exponential would go to zero faster than the increase in the polynomial term.

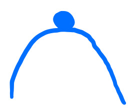
- ~ Lyapunov Stability

  The solution $x=0$ is stable if $\forall \epsilon > 0 \ \exists \delta > 0$ such that $|x(0) - 0| < \delta \Rightarrow |e^{At}x(0) - 0| < \epsilon \ \forall t > 0$



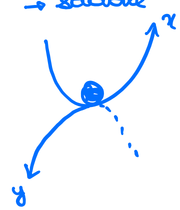
$\dot{x} = -x$

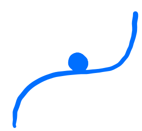
is minimum of this well stable?
+friction ⇒ stable
−friction ⇒ oscillation, stable?



$\dot{x} = +x$

not stable
$\lambda^2 - 1 = 0$
$\lambda = \pm 1 \to e^{+t}, e^{-t} \to$ "saddle"





$\dot{x} = -x^2$
$x = 0, \ \dot{x} = 0$
$x = \epsilon > 0, \ \dot{x} < 0$
$x = -\epsilon < 0, \ \dot{x} < 0$

-

Bicycle : An Unstable System ?

- Model

forward velocity

Given
$$\begin{cases} M\underset{\begin{bmatrix}\dot x_3\\ \dot x_4\end{bmatrix}}{\ddot q} + v C_1 \underset{\begin{bmatrix}x_3\\ x_4\end{bmatrix}}{\dot q} + \left(v^2 K_2 + g K_0\right) \underset{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}{q} = f, \quad q = \begin{bmatrix}\phi\\ \delta\end{bmatrix}, \quad f = \begin{bmatrix}T_\phi\\ T_\delta\end{bmatrix} \\ \phi : \text{balance angle},\ \delta : \text{steer angle},\ f : \text{input} \end{cases}$$

Set $x_1 = \phi$, $x_2 = \delta$, $x_3 = \dot\phi$, $x_4 = \dot\delta$

$$\Rightarrow \underbrace{\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\end{bmatrix}}_{\dot x} \;=\; \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \hline \multicolumn{2}{c|}{-M^{-1}\times} & \multicolumn{2}{c}{-M^{-1}\times}\\ \multicolumn{2}{c|}{[gK_0+v^2K_2]} & \multicolumn{2}{c}{vC_1} \end{array}\right]}_{A} \underbrace{\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}}_{x}$$

$$M = \begin{bmatrix}80.8 & 2.3\\ 2.3 & 0.3\end{bmatrix},\ g = 9.8,\ v = 2$$

$$C_1 = \begin{bmatrix}0 & 33.9\\ -0.9 & 1.7\end{bmatrix},\ K_0 = \begin{bmatrix}-81 & -2.6\\ -2.6 & -0.8\end{bmatrix}$$

$$K_2 = \begin{bmatrix}0 & 76.6\\ 0 & 2.7\end{bmatrix}$$

PARAMETERS

$$\Rightarrow A = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 9.58 & -4.13 & -0.22 & -0.67\\ 12.25 & 21.6 & 7.7 & -6.28\end{bmatrix}$$

Would $[x_1\ x_2\ x_3\ x_4] = [0\ 0\ 0\ 0]$ be stable?

$\mathrm{Re}\{\lambda\} > 0 \Rightarrow$ unstable

- Eigenvalues $\simeq -8.6,\ -3.1,\ 2.6 \pm 1.7j$

Unstable eigenvalues are complex. What does this indicate?

- If $x(0) = 0 \Rightarrow x(t) = e^{At}x(0) = 0$. TRUE or FALSE.

For small perturbations around $x = 0$, does it come back to $x = 0$? NO.

- "Self-stability" for larger v.