Control : Why do we need **it** ?

What are the design specifications ?

- phase margin
    - ↳ robustness to disturbances / uncertainties
    - ↳ some transient response specification?

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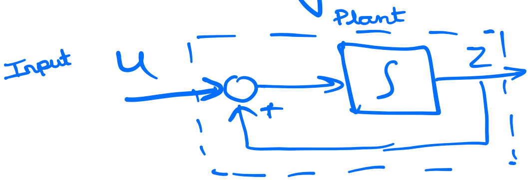
Control : Why do we need it ?

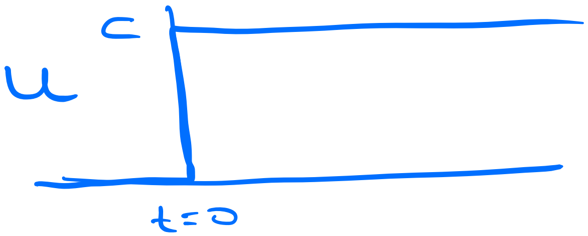
What are the design specifications ?

- phase margin
  - ↳ $\boxed{\text{robustness}}$ to disturbances / uncertainties
  - ↳ some transient response specification?
- to reduce errors that come in loops
  - ↳ desired - actual (output/states)
  - ⇒ use control to make system reach desired state



- System $\dot{z} = z + u$, $z(0) = 0$

Goal: design input u so that $z(1) = 10$.



$c = \frac{10}{e-1}$

- $\dot{z} = z + u$, $u = 0 \Rightarrow z = z(0) = 0$

Note $z = 0$ is not a stable solution because (FILL IN THIS BLANK) ______________________.

- Try u = constant, c $(\neq 0)$ after $t = 0$.

$$\Rightarrow \dot{z} = z + c$$

$$\Rightarrow \dot{z} - z = c$$

$$e^{-t} \times [\quad \Rightarrow \frac{d\, z e^{-t}}{dt} = c e^{-t}$$

$$\int \quad \Rightarrow z(t) e^{-t} - \underset{=0}{\cancel{z(0)}} = \frac{c}{(-1)} \left(e^{-t} - 1 \right)$$

$$\Rightarrow \quad z(t) = c\,(e^{t} - 1)$$

We want $z(1) = 10 \Rightarrow \quad c = \dfrac{10}{e-1}$

- Other u's also possible. This is only for a class of constant 'u'.

- Before searching for a 'u', it would be nice to know if a 'u' exists.

- Feedback loop: How to design 'u'?

$\rightarrow$ Kalman Filter

Kalman's Example

$\text{rank}[B \;\; AB \;\; A^2B \;\; A^3B] = 2\,?$

states: x_1, x_2, x_3, x_4

input

- $\dot{x}_1 = 2x_1 + 3x_2 + 2x_3 + x_4 + u$

  $\dot{x}_2 = -2x_1 - 3x_2 - 2u$

  $\dot{x}_3 = -2x_1 - 2x_2 - 4x_3 + 2u$

  $\dot{x}_4 = -2x_1 - 2x_2 - 2x_3 - 5x_4 - u$

$$A = \begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix}$$

- $\dot{x} = Ax + Bu$

$$B = [1 \;\; -2 \;\; 2 \;\; -1]'$$

- Diagonalize A {eigenvalues, eigenvectors}

  $\{\lambda_1, v_1\}$ $\{\lambda_2, v_2\}$, $\{\lambda_3, v_3\}$, $\{\lambda_4, v_4\}$

  -1 $\quad -2$ $\quad -3$ $\quad -4$

  $\Rightarrow AV = VD$, $V = [v_1 \;\; v_2 \;\; v_3 \;\; v_4\}$

- $z = V^{-1}x$, $\quad \dot{z} = V^{-1}\dot{x} = V^{-1}Ax + V^{-1}Bu$

  $$\Rightarrow \dot{z} = \underbrace{V^{-1}AV}_{D = \text{diag}\{-1,-2,-3,-4\}} z + V^{-1}Bu$$

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

rank $[B \ \ AB \ \ A^2B \ \ A^3B] = ?$

2

$$\equiv \quad \dot{z}_1 = -z_1 + u$$

$$\dot{z}_2 = -2z_2 \longleftarrow \text{no input}$$

$$\dot{z}_3 = -3z_3 + u$$

$$\dot{z}_4 = -4z_4 \longleftarrow \text{no input}$$

Because there are no inputs to z_2, z_4, we cannot reach arbitrary z_2, z_4, and accordingly in the x-space.

- uncontrollable systems? Bicycle?

Controllability

The system $\dot{x} = Ax + Bu$ is controllable if there is an input $u(t)$, $0 \leq t \leq T$ such that it can drive the system from $x(t=0) = 0$ to any desired state $x(t=T)$ in finite time T.

- Solution of $\dot{x} = Ax + Bu$

$$y = Cx + Du$$

$x \in \mathbb{R}^{n\times1}$

$u \in \mathbb{R}^{m\times1}$

$$\dot{[\ ]}_{n\times1} = [\ \ ]_{n\times n}[\ ]_{n\times1} + [\ \ ]_{n\times m}[\ ]_{m\times1}$$

$$\dot{x} = Ax + Bu$$

$$\Rightarrow \dot{x} - Ax = Bu$$

$e^{-At} \times [$

$$\Rightarrow e^{-At}\dot{x} + \underbrace{e^{-At}(-A)}_{\frac{d}{dt}e^{-At}}x = e^{-At}Bu$$

$$\Rightarrow \frac{d}{dt}e^{-At}x = e^{-At}Bu$$

$\int_0^t$

$$\Rightarrow e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$\Rightarrow x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{convolution integral}}$$

$x(0)=0$ for controllability

$$\Rightarrow \quad x(t) = \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

At finite time $t=T$

$$x(T) = \int_0^T e^{A(T-\tau)} B\, u(\tau)\, d\tau$$

What all $x(T)$ can be reached?

$$e^{A(T-\tau)} = I + A(T-\tau) + A^2 \frac{(T-\tau)^2}{2!} + \dots$$

- Is this series going on and on ...

Cayley-Hamilton Theorem: If $\overset{=\det(\lambda I - A)}{f(\lambda)}=0$ is the characteristic polynomial of A, then $f(A)=0$

$$\Big\{ A = VDV^{-1} \Rightarrow A^k = \underbrace{VD\underbrace{V^{-1}V}D\underbrace{V^{-1}\cdots}\cdots\underbrace{V}DV^{-1}}_{k\ \text{times}}$$
$$= V D^k V^{-1} \Big\}$$

- $A^n + \alpha_1 A^{n-1} + \dots + \alpha_n I = 0 \Rightarrow A^n = -\alpha_1 A^{n-1} - \dots - \alpha_n I$

$$\Rightarrow e^{A(T-\tau)} = \overset{\curvearrowright (T-\tau)^k \ldots,\ \alpha_i \ldots}{f_0(T-\tau)}\, I + f_1(T-\tau)\, A + \cdots + f_{n-1}(T-\tau)\, A^{n-1}$$

$$\times B \Rightarrow e^{A(T-\tau)} B = f_0(T-\tau)\, B + f_1(T-\tau)\, AB + \cdots + f_{n-1}(T-\tau)\, A^{n-1} B$$

$$= \underset{n \times mn}{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}} \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}_{mn \times}$$

$$\underset{n\times 1}{\dot{x}} = \underset{n\times n}{A}\,\underset{n\times 1}{x} + Bu$$

$$\Rightarrow \underset{n\times n}{e^{A(T-\tau)}} = \underset{\text{scalar}}{f_0(T-\tau)}\, I + \underbrace{\underset{n\times n,\ \text{scalar}}{f_1(T-\tau)}\, \underset{n\times n}{A}}_{n\times n} + \dots + f_{n-1}(T-\tau)A^{n-1}$$

$$\times B \Rightarrow e^{A(T-\tau)}B = f_0(T-\tau)B + f_1(T-\tau)AB + \dots + f_{n-1}(T-\tau)A^{n-1}B$$

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$$= \underset{n\times mn}{\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}} \underset{mn\times m}{\begin{bmatrix} f_0(T-\tau)I_{m\times m} \\ f_1(T-\tau)I_{m\times m} \\ \vdots \\ f_{n-1}(T-\tau)I_{m\times m} \end{bmatrix}}$$

Check for

$m = 1$

$m = 2$

...

$f_0 \rightarrow m\times 1$? $f_0 \underset{n\times m}{B}$

↳ scalar

dimensionality?

$$x(T) = \int_0^T e^{A(T-\tau)} B\, u(\tau)\, d\tau$$

$$\Rightarrow x(T) = \int_0^T [B \;\; AB \;\; \ldots \;\; A^{n-1}B] \begin{bmatrix} f_0(T-\tau)\, I \\ f_1(T-\tau)\, I \\ \vdots \\ f_{n-1}(T-\tau)\, I \end{bmatrix} u(\tau)\, d\tau .$$

$$\Rightarrow x(T) = \underbrace{[B \;\; AB \;\; \ldots \;\; A^{n-1}B]}_{W_c} \int_0^T \begin{bmatrix} f_0(T-\tau)\, I \\ f_1(T-\tau)\, I \\ \vdots \\ f_{n-1}(T-\tau)\, I \end{bmatrix} u(\tau)\, d\tau$$

controllability matrix

(~~$n\; m \times m$~~)

$n \times nm$

controllability

($x(T)$ can be anywhere)

$\Rightarrow$

or

$\Leftarrow$?

$\text{rank}(W_c) = n$

Claim: Controllability $\Rightarrow$ rank$(W_c) = n$.

Suppose rank$(W_c) < n$.

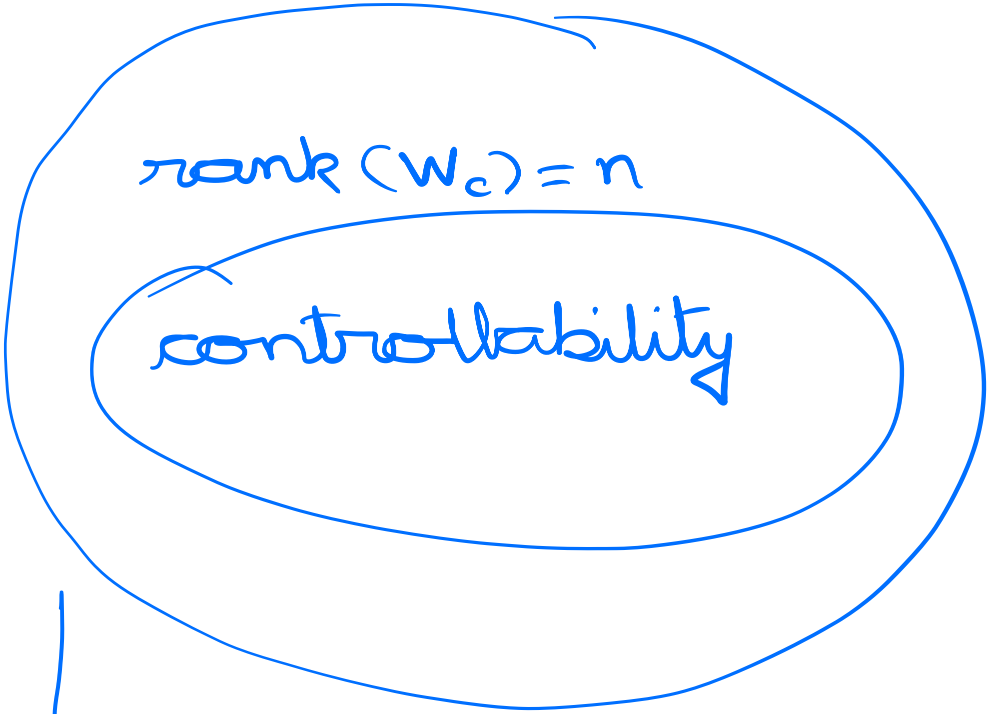
$\Rightarrow$ Rangespace(W_c) is not the whole of the vector space

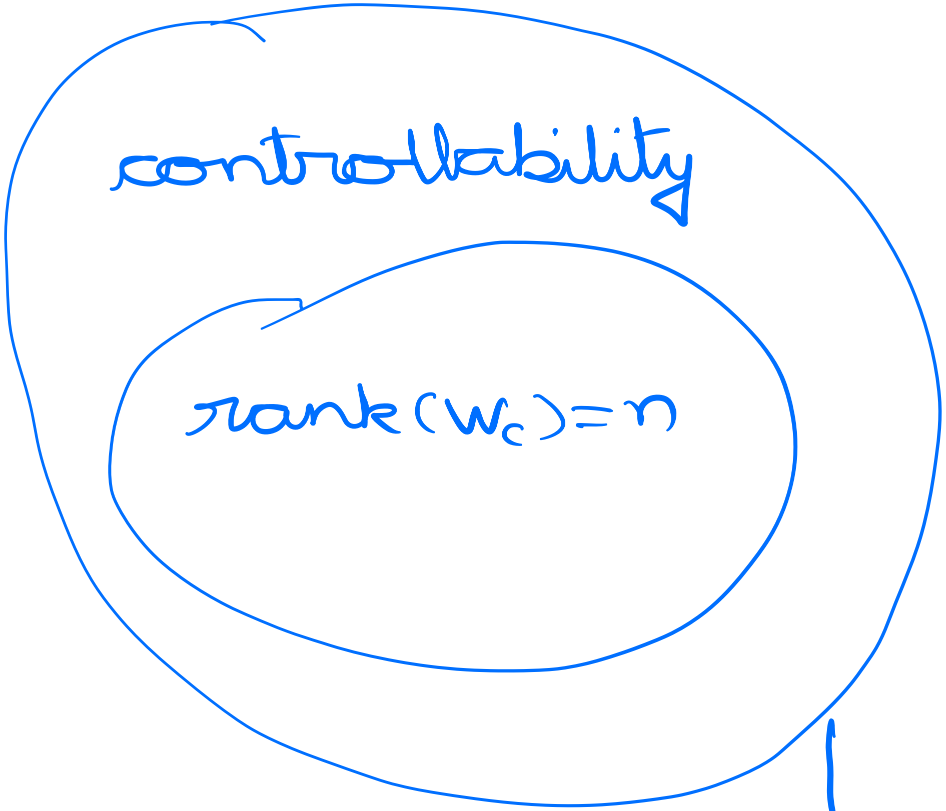
$\Rightarrow$ Desired $x(T)$ in vector space $\overset{\text{"minus"}}{\setminus}$ Rangespace(W_c)

cannot be reached.

$\Rightarrow$ Contradiction! as given that the system is controllable.

$\Rightarrow$ rank$(W_c) = n$.





Which picture represents

controllability $\Rightarrow$ rank $(W_c) = n$.

→ ✓

Want to prove ←

Claim: rank$(W_c) = n \Rightarrow$ Controllability

Given $x(T)$, there should be an input $u(\tau)$ $0 \le \tau \le T$

How to find this $u(\tau)$?

$$x(T) = \int_0^T e^{A(T-\tau)} B \, u(\tau) \, d\tau$$

transpose →

$$\text{Define } u(\tau) = B' e^{A'(T-\tau)} \left(\underbrace{\int_0^T e^{A(T-\tau)} B B' e^{A'(T-\tau)} d\tau}_{\text{Grammian}} \right)^{-1} x(T)$$

$$\uparrow$$

$$\text{Note: } \int_0^T e^{A(T-\tau)} B \; B' e^{A'(T-\tau)} \left(\int_0^T e^{A(T-\tau)} B B' e^{A'(T-\tau)} d\tau \right)^{-1} x(T) \, d\tau$$

$$= \int_0^T e^{A(T-\tau)} BB' e^{A'(T-\tau)} \left(\int_0^T e^{A(T-\tau)} BB' e^{A'(T-\tau)} d\tau \right)^{-1} x(T) \, d\tau$$

$$= \left(\int_0^T e^{A(T-\tau)} BB' e^{A'(T-\tau)} d\tau \right) \left(\int_0^T e^{A(T-\tau)} BB' e^{A'(T-\tau)} d\tau \right)^{-1} x(T)$$

inverses

$$= x(T)$$

This $u(\tau)$ assumes that the Grammian $\int_0^T e^{A(T-\tau)} B B' e^{A'(T-\tau)} d\tau$ is invertible

Suppose it is not invertible

$\Rightarrow \exists \, v \neq 0$ such that

there exists

$$\int_0^T e^{A(t-\tau)} B B' e^{A'(T-\tau)} d\tau \cdot v = 0$$

$$v' \times [\Rightarrow v' \int_0^T e^{A(T-\tau)} BB' e^{A'(T-\tau)} d\tau \cdot v = 0$$

$$\Rightarrow \int_0^T \underbrace{v' e^{A(T-\tau)}}_{z'} B \cdot \underbrace{B' e^{A'(T-\tau)} v}_{\substack{m\times n \;\; n\times n \;\; n\times 1 \\ = z_{m\times 1}}} \, d\tau = 0$$

$$\Rightarrow \int_0^T z' z \, d\tau = 0 \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}, \quad z'z = z_1^2 + z_2^2 + \dots + z_m^2$$

$$\Rightarrow z = 0 = z'$$ because integral$(\Sigma(+ve)) = 0$.

$$\Rightarrow v' e^{A(T-\tau)} B = 0 \qquad 0 \le \tau \le T$$

differentiate w.r.t τ

$$\Rightarrow v' e^{A(T-\tau)} \cdot A B = 0 \qquad 0 \le \tau \le T$$

differentiate w.r.t τ

$$\Rightarrow v' e^{A(T-\tau)} A^2 B = 0 \qquad 0 \le \tau \le T$$

. . . (n-1) times

$$\Rightarrow v' e^{A(T-\tau)} A^{n-1} B = 0 \qquad 0 \leq \tau \leq T$$

$$\Rightarrow \underbrace{v' e^{A(T-\tau)}}_{\text{non-zero vector}} \underbrace{[B \quad AB \quad \ldots \quad A^{n-1}B]}_{W_c} = 0$$

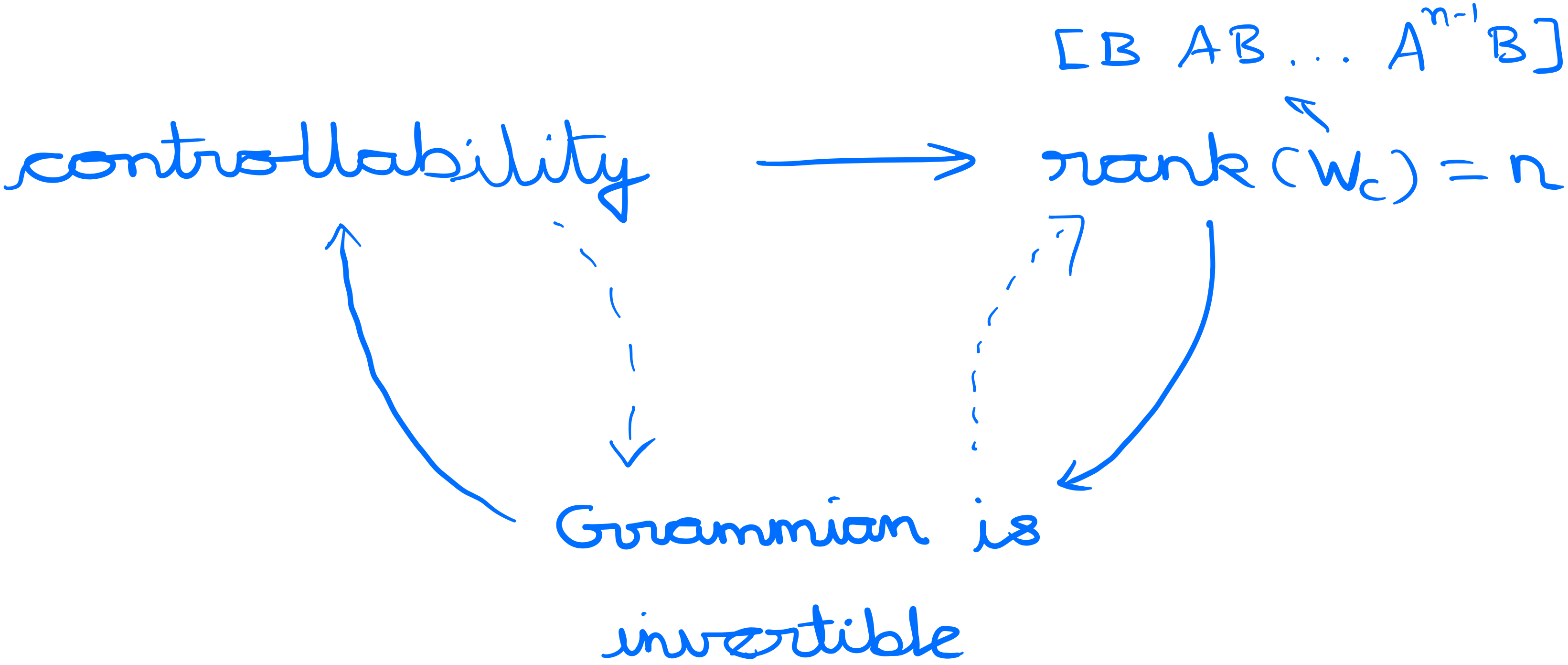
This should not be possible because

rank $(W_c) = n$.

$\Rightarrow$ Grammian is invertible

$\Rightarrow$ $u(\tau)$ can be defined as above

$\Rightarrow$ System is controllable.



- These three are equivalent
- Advantage : algebraic test for controllability

Bicycle : Controllable ?

Is rank $[B \ \ AB \ \ A^2B \ \ A^3B] = 4$?

- Model

Given $\begin{cases} M\ddot{q} + vC_1\dot{q} + (v^2K_2 + gK_0)q = f, \quad q = \begin{bmatrix}\phi\\ \delta\end{bmatrix}, \quad f = \begin{bmatrix}T_\phi\\ T_\delta\end{bmatrix} \\ \phi : \text{balance angle}, \ \delta : \text{steer angle}, \ f : \text{input}\end{cases}$

Set $x_1 = \phi, \ x_2 = \delta, \ x_3 = \dot{\phi}, \ x_4 = \dot{\delta}$

$$\Rightarrow \underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}}_{\dot{x}} = \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \multicolumn{2}{c|}{-M^{-1}\times} & \multicolumn{2}{c}{-M^{-1}\times} \\ \multicolumn{2}{c|}{[gK_0+v^2K_2]} & \multicolumn{2}{c}{vC_1}\end{array}\right]}_{A} \underbrace{\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}}_{x} + \underbrace{\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline \multicolumn{2}{c}{M^{-1}}\end{array}\right]}_{B} \underbrace{\begin{bmatrix}T_\phi\\ T_\delta\end{bmatrix}}_{u}$$

$$M = \begin{bmatrix}80.8 & 2.3\\ 2.3 & 0.3\end{bmatrix}, \ g = 9.8, \ v = 2$$

$$C_1 = \begin{bmatrix}0 & 33.9\\ -0.9 & 1.7\end{bmatrix}, \ K_0 = \begin{bmatrix}-81 & -2.6\\ -2.6 & -0.8\end{bmatrix}$$

$$K_2 = \begin{bmatrix}0 & 76.6\\ 0 & 2.7\end{bmatrix}$$

PARAMETERS

$$\Rightarrow A = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 9.58 & -4.13 & -0.22 & -0.67\\ 12.25 & 21.6 & 7.7 & -6.28\end{bmatrix}, \ B = \left[\begin{array}{c|c}0 & 0\\ 0 & 0\\ 0.016 & -0.122\\ -0.122 & 4.28\end{array}\right]$$

$$\text{rank}([B \; AB \; \dots \; A^{n-1}B]) = n \iff \text{rank}([\tilde{B} \; \tilde{A}\tilde{B} \; \dots \; \tilde{A}^{n-1}\tilde{B}]) = n$$

$$\left\{ \tilde{A} = T^{-1}AT, \; \tilde{B} = T^{-1}B \right\}$$

tilde 01.09.2022

1. $\operatorname{rank}([B \ AB \ \ldots A^{n-1}B]) = n \iff \operatorname{rank}([\tilde{B} \ \tilde{A}\tilde{B} \ \ldots \ \tilde{A}^{n-1}\tilde{B}]) = n$

$$\left\{ \tilde{A} = \underset{TAT^{-1}}{\cancel{T^{-1}AT}}, \quad \tilde{B} = \underset{TB}{\cancel{T^{-1}B}} \right\}$$

2. $\underline{(A, B)}$ is controllable $\iff$ $\underline{(TAT^{-1}, TB)}$ is controllable

3. Controllability is independent of any (invertible) co-ordinate transformations.

$$\dot{x} = Ax + Bu \qquad z = Tx, \ T \text{ is invertible}$$

$$\dot{z} = T\dot{x} = T(Ax + Bu) = \underbrace{TAT^{-1}}_{\tilde{A}} z + \underbrace{TB}_{\tilde{B}} u$$

$\Rightarrow$ We have to show #2

$$W_c = [B \;\; AB \;\; \dots \;\; A^{n-1}B]$$

$$\tilde{W}_c = [\tilde{B} \;\; \tilde{A}\tilde{B} \;\; \dots \;\; \tilde{A}^{n-1}\tilde{B}] \qquad = T W_c$$

$$= [TB \;\; TA\underbrace{T^{-1}T}B \;\; \dots \;\; \underbrace{TA\underbrace{T^{-1}T}A\underbrace{T^{-1}\dots T}A\underbrace{T^{-1}T}B}_{n-1 \text{ times}}]$$

$$= [TB \;\; TAB \;\; \dots \; TA^{n-1}B]$$

check?

$$= T W_c$$

$\tilde{W}_c = T W_c$ or $T^{-1}\tilde{W}_c = W_c$

as T is invertible, $\text{rank}(W_c) = \text{rank}(\tilde{W}_c)$.

$\Rightarrow$ Above statements 1, 2, & 3.

The Controllable & The Uncontrollable

- Some parts of the state space may be controllable, some may not be controllable (uncontrollable)
- $\dot{x} = Ax + Bu$

  $W_c = [B \ AB \ \ldots \ A^{n-1} B]$

  Suppose $\operatorname{rank}(W_c) = r < n$

  $\Rightarrow$ dimension$\{$Rangespace$(W_c)\} = r < n$

Recall: Rangespace of a "long" matrix such as W_c is the span of its columns.

Let B_c be a basis for the Rangespace(W_c)

$$B_c = \{ v_1, v_2, \dots v_r \}$$

Extend this to a basis B of the entire state space,

$$B = \{ \underbrace{v_1, v_2, \dots v_r}_{\substack{r \\ \text{from } B_c}}, \underbrace{v_{r+1}, \dots v_n}_{n-r} \}$$

$$A \overbrace{[v_1\ v_2 \dots v_r\ v_{r+1} \dots v_n]}^{T\text{, which is invertible}}$$

$$= [Av_1\ Av_2\ Av_r\ Av_{r+1} \dots Av_n]$$

v_1 = linear combination of columns of W_c

$= \sum_i \alpha_i \times$ columns of W_c

$A v_1 = \sum_i \alpha_i \times A \times$ columns of W_c → $[B \;\; AB \ldots A^{n-1} B]$

$= \sum_i \alpha_i \times$ columns of AW_c → $[AB \;\; A^2 B \ldots A^n B]$

Cayley Hamilton Theorem

$$A^n = \sum_{i=0}^{n-1} r_i A^i$$

$= \sum_i \beta_i \times$ columns of W_c

$=$ linear combination of $\{v_1, \ldots v_r\}$

In fact

$A v_i \; (1 \le i \le r) =$ linear combination of $\{v_1, \ldots v_r\}$

But

$A v_i \; (r+1 \le i \le n) =$ linear combination of $\{v_1, \ldots v_r, v_{r+1}, \ldots v_n\}$

$$\overset{n\times n}{A}\,\overbrace{[v_1 \mid v_2 \mid \cdots \mid v_r \mid v_{r+1} \mid \cdots \mid v_n]}^{T}_{n\times n}$$

$$= [Av_1 \mid Av_2 \mid \cdots \mid Av_r \mid Av_{r+1} \mid \cdots \mid Av_n]_{n\times n}$$

$$= \underbrace{[v_1 \mid v_2 \mid \cdots \mid v_r \mid v_{r+1} \mid \cdots \mid v_n]}_{T_{n\times n}} \left[\begin{array}{ccc} \multicolumn{3}{l}{\overbrace{\hspace{4em}}^{r}} \\ \delta_{11} & \delta_{21} & \\ \delta_{12} & \delta_{22} & \\ \vdots & \vdots & \\ \delta_{1r} & \delta_{2r} & \\ \hline 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array}\right]_{n\times n} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} r \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} n-r \end{array}$$

$$Av_1 = \delta_{11}v_1 + \delta_{12}v_2 + \cdots + \delta_{1r}v_r + 0v_{r+1} + \cdots\ 0v_n$$

$$Av_2 = \delta_{21}v_1 + \delta_{22}v_2 + \cdots + \delta_{2r}v_r + 0v_{r+1} + \cdots\ 0v_n$$

$$Av_{r+1} = \delta_{r+1,1}v_1 + \delta_{r+2,2}v_2 + \cdots + \delta_{r+1,r}v_r + \delta_{r+1,r+1}v_{r+1} + \cdots + \delta_{r+1,n}v_n$$

$$AT = T \begin{array}{c|cc|c} & r & n-r & \\ \hline & \tilde{A}_{11} & \tilde{A}_{12} & \}\, r \\ & 0 & \tilde{A}_{22} & \}\, n-r \end{array}$$

$\times T^{-1}$

$$A = T \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} T^{-1}$$

$[B \;\; AB \ldots A^{n-1}B]$

Do the columns of $B \in$ Rangespace(W_c)?

Yes, it is in their span

$\Rightarrow$ B is a linear combination of $\{v_1, \ldots v_r\}$

$$\underset{n \times m}{B} = \underset{n \times n}{\overbrace{[v_1 \;\; v_2 \ldots v_r \;\; v_{r+1} \ldots v_n]}^{T}} \begin{array}{c|c} \overset{r \times m}{\tilde{B}_1} & \}\, r \\ \hline \underset{(n-r) \times m}{0} & \}\, n-r \end{array}$$

Use $B = [b_1 \ b_2 \ \dots b_m]_{n \times m}$, $b_1, b_2, \dots b_m$

$b_1 = \theta_{11} v_1 + \theta_{12} v_2 + \dots \theta_{1r} v_r + 0 . v_{r+1} \dots 0 . v_n$

Similarly for $b_2, \dots b_m$.

Theorem: Suppose for a matrix pair (A, B), $W_c = [B\ AB\ \dots\ A^{n-1}B]$ and $\text{rank}(W_c) = r < n$. Then there exists a state transformation T such that

controllable

a) $$T^{-1}AT = \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \underbrace{0}_{r} & \underbrace{\tilde{A}_{22}}_{n-r} \end{array}\right] \begin{array}{l} \}\, r \\ \}\, n-r \end{array}, \quad T^{-1}B = \left[\begin{array}{c} \tilde{B}_1 \\ \hline 0 \end{array}\right] \begin{array}{l} \}\, r \\ \}\, n-r \end{array}$$

b) $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable.

Check this.

What is the rank of $[\tilde{B}_1\ \ \tilde{A}_{11}\tilde{B}_1\ \dots\ \tilde{A}_{11}^{r-1}\tilde{B}_1]$?

Quiz 2

01.09.2022

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix}$$

$$\left.\begin{array}{l} \text{a. eigenvalues ?} \\ \text{b. eigenvectors ?} \\ \text{c. } e^{At} \text{ ?} \end{array}\right\} \quad \begin{array}{l} \lambda \neq \mu \\ \lambda, \mu \text{ are constants} \end{array}$$

d. $\lim\limits_{\mu \to \lambda} e^{At}$?

e. Sketch eigenvectors as $\mu \to \lambda$.

(treat as vectors in a plane)

show for some (≥ 3) values

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u$$

05.09.2022

Design u to reach state $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

- Solve for $u \rightarrow x$. Then "backcalculate" the u needed for $x(\text{final}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

↳ Given final state, what input is needed

$$x(T) = \int_0^T e^{A(T-\tau)} B \, u(\tau) \, d\tau + e^{AT} x(0)$$

transpose →

$$u(\tau) = B' e^{A'(T-\tau)} \underbrace{\left(\int_0^T e^{A(T-\tau)} B B' e^{A'(T-\tau)} \, d\tau \right)^{-1}}_{\text{Grammian}} \left(x(T) - e^{AT} x(0) \right)$$

This is one way to construct u.

But, Grammian needs to be invertible

How to check this?

↳ "Feedforward" → precalculate u

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ origin is stable?

unstable

eigenvalues: $\det(\lambda I - A) = 0$

↳ any real number

$$\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix} \xrightarrow{\det} (\lambda-1)^2 = 0 \xrightarrow{e.v} 1, 1 \in RHP$$

$u = -Kx$, $K = [k_1 \;\; k_2]$, state feedback

$$\Rightarrow \dot{x} = Ax + Bu = Ax + B(-Kx)$$

$$\dot{x} = (A - BK)\,x$$

Feedback can be used to change the dynamics . . .

$$A - BK = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

What are eigenvalues?

$$= \begin{bmatrix} 1-k_1 & -k_2 \\ -k_1 & 1-k_2 \end{bmatrix}$$

$$\det(sI - (A-BK)) = 0$$

$$\begin{vmatrix} s-(1-k_1) & k_2 \\ k_1 & s-(1-k_2) \end{vmatrix} = (s-(1-k_1))(s-(1-k_2) - k_1 k_2$$

$$= s^2 - (2-k_1-k_2)s + (1-k_1-k_2)$$

$$= s^2 - (1+1-k_1-k_2)s + 1.(1-k_1-k_2)$$

$$= (s-1)(s-(1-k_1-k_2))$$

eigenvalues : 1, $1-k_1-k_2$
(dynamics)

not all eigenvalues can be designed

- Controllable ?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{rank}\left\{ \begin{bmatrix} B & AB \end{bmatrix} \right\}$$

$$= \text{rank}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} = 1$$

Design of Control

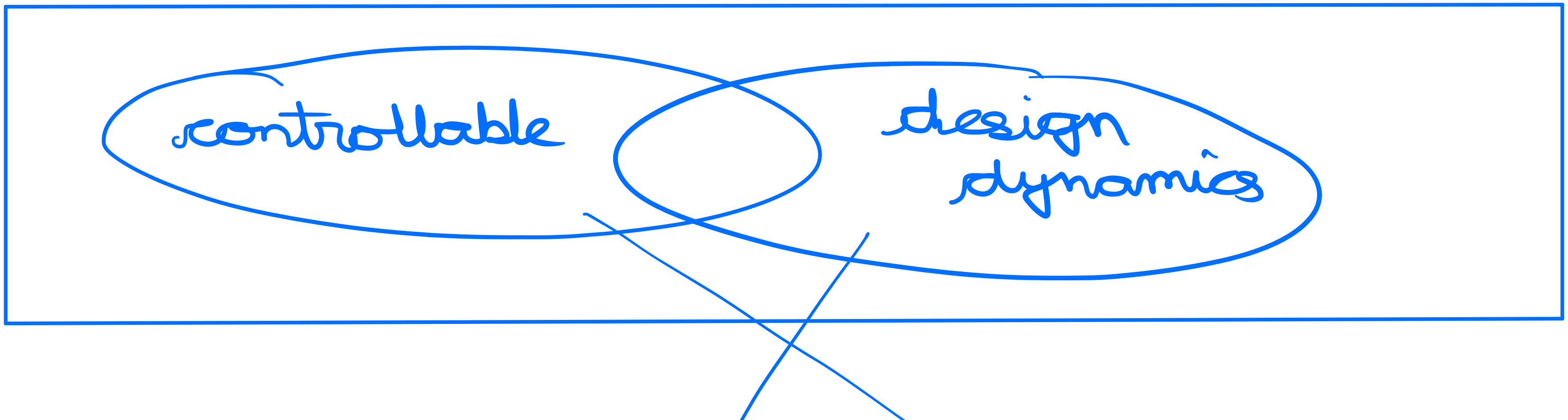
- Check Controllability

- "Feedforward" $\int \quad (\quad)^{-1} dz(x(T))$

or

- "Feedback" $u = -Kx$

  to change dynamics



Design any dynamics $\Longleftrightarrow$ Controllable

THEOREM

Eigenvalues of $A-BK$ are freely assignable $\Longleftrightarrow$ (A, B) is controllable

$$\dot{x} = Ax + Bu, \quad u = -Kx$$

$$\Rightarrow \dot{x} = (A - BK)x$$

Multiple solutions possible

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u \quad , \quad u = -K\,x$$

eigenvalues of A: $\det\left\{ \begin{bmatrix} s & -1 \\ -\Omega^2 & s \end{bmatrix} \right\} = 0$

$$s = \pm j\Omega$$

arbitrary place eigenvalues?

controllability? $\operatorname{rank}\left\{ \begin{bmatrix} B & AB \end{bmatrix} \right\}$

$\operatorname{rank}\left\{ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \Omega^2 & 0 \end{bmatrix} \right\} = 2 \Rightarrow$ controllable

$\therefore$ We should be able to place eigenvalues anywhere.

$$A - BK = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -k_{11} & 1-k_{12} \\ \Omega^2 - k_{21} & -k_{22} \end{bmatrix}$$

$$\det\{ sI - (A-BK) \} = 0$$

$$\Rightarrow \begin{vmatrix} s + k_{11} & -1 + k_{12} \\ -\Omega^2 + k_{21} & s + k_{22} \end{vmatrix} = 0$$

$$\Rightarrow (s + k_{11})(s + k_{22}) - (-\Omega^2 + k_{21})(-1 + k_{12}) = 0$$

$$\Rightarrow s^2 + (k_{11} + k_{22})s + k_{11}k_{22} - (\Omega^2 - k_{21} - k_{12}\Omega^2 + k_{21}k_{12}) = 0$$

$$\Rightarrow s^2 + (k_{11} + k_{22})s + k_{11}k_{12} - k_{21}k_{12} - (1 - k_{12})\Omega^2 + k_{21} = 0$$

desired eigenvalues (λ_1, λ_2)

$$\Rightarrow (s-\lambda_1)(s-\lambda_2) = 0$$

$$\Rightarrow s^2 - (\lambda_1+\lambda_2)s + \lambda_1\lambda_2 = 0$$

$$\Rightarrow \quad -(\lambda_1+\lambda_2) = k_{11}+k_{22}$$

$$\lambda_1\lambda_2 = k_{11}k_{12} - k_{21}k_{12} - (1-k_{12})\Omega^2 + k_{21}$$

2 equations, 4 unknowns

$\Rightarrow$ multiple solutions exist,

which one to choose?

Bicycle : Control Input

· Model

Given $\begin{cases} M\ddot{q} + vC_1\dot{q} + (v^2K_2 + gK_0)q = f, \quad q = \begin{bmatrix}\phi \\ \delta\end{bmatrix}, \quad f = \begin{bmatrix}T_\phi \\ T_\delta\end{bmatrix} \\ \phi : \text{balance angle, } \delta : \text{steer angle, } f : \text{input} \end{cases}$

Set $x_1 = \phi, \; x_2 = \delta, \; x_3 = \dot{\phi}, \; x_4 = \dot{\delta}$

$$\Rightarrow \underbrace{\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4\end{bmatrix}}_{\dot{x}} \underset{=}{=} \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \multicolumn{2}{c|}{-M^{-1}\cdot[gK_0 + v^2K_2]} & \multicolumn{2}{c}{-M^{-1}vC_1} \end{array}\right]}_{A} \underbrace{\begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix}}_{x} + \underbrace{\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline \multicolumn{2}{c}{M^{-1}} \end{array}\right]}_{B} \underbrace{\begin{bmatrix}T_\phi \\ T_\delta\end{bmatrix}}_{u}$$

· MATLAB

→ rank(ctrb(A, B))

→ place(A, B, p)

desired eigenvalues

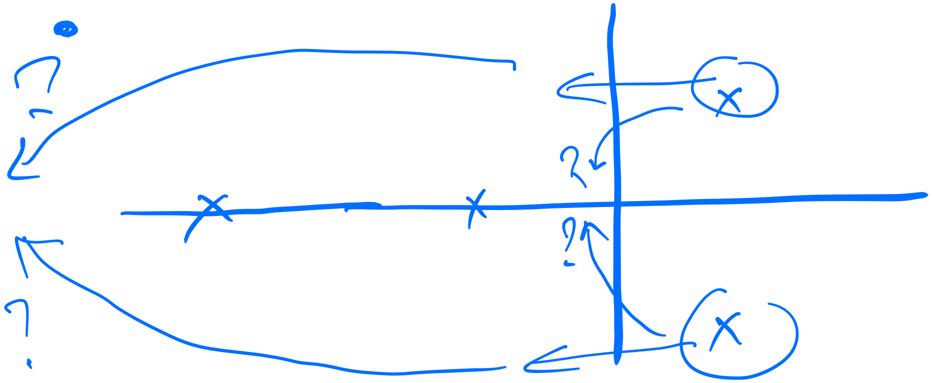
how would we choose?

Where to $\underbrace{\text{place eigenvalues}}_{\text{dynamics}}$?

Bicycle model

Eigenvalues $\approx -8.6, -3.1, 2.6 \pm 1.7j$

What eigenvalues could we choose?



Issues to consider:

- large (negative) desired eigenvalues

  ⇒ large K

  ⇒ saturation of input / actuator

- others.

Companion Matrix

↓

Single Input

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$\cancel{\det(A)} = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$$

$$\det(sI-A) = \begin{vmatrix} s+a_1 & a_2 & \cdots & a_{n-1} & a_n \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{vmatrix}$$

$$n=2 \qquad \begin{vmatrix} s+a_1 & a_2 \\ -1 & s \end{vmatrix} = s^2 + a_1 s + a_2$$

$$n=3 \qquad \begin{vmatrix} s+a_1 & a_2 & a_3 \\ -1 & s & 0 \\ 0 & -1 & s \end{vmatrix} = (s+a_1)\begin{vmatrix} s & 0 \\ -1 & s \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ -1 & s \end{vmatrix}$$

$$= s^3 + a_1 s^2 + a_2 s + a_3$$

Companion Form

- $\dot{x} = Ax + Bu$, $A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$
- $\det(A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$
- $\det(A - BK) = s^n + (a_1 + k_1) s^{n-1} + (a_2 + k_2) s^{n-2} + \dots + (a_n + k_n]$

$K = [k_1\ k_2 \dots k_n]$

$u = -Kx$

$k_1 = b_1 - a_1$

$k_2 = b_2 - a_2$

$k_n = b_n - a_n$

$$s^n + b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n$$

- Desired polynomial: $(s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_n)$
- Desired dynamics: $\underbrace{\lambda_1, \lambda_2, \dots \lambda_n}_{\text{eigenvalues}}$

Companion form exists $\Leftrightarrow$ Controllable.

THEOREM

(A, B) is controllable $\Leftrightarrow$ there is an invertible transformation T that transforms (A, B) into the companion form,

$$\underset{A_c}{TAT^{-1}} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \underset{B_c}{TB} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

($\Leftarrow$)

Proof: Such an invertible transformation does not change controllability

(A_c, B_c) is controllable $\Leftrightarrow$

— (A, B) is controllable

How to show (A_c, B_c) is controllable?

$$\text{rank}\left(\left[B_c \;\; A_cB_c \;\cdots\; A_c^{n-1}B_c\right]\right) = n\,?$$

diagonal

$$\begin{array}{cccccc} \left[\;1\right. & -a_1 & a_1^2-a_2 & & & \\ 0 & 1 & -a_1 & & & \\ \vdots & \vdots & 1 & & & \\ 0 & 0 & \vdots & 1 & & \\ & & 0 & & 1 & \left.1\;\right] \\ \uparrow & \uparrow & \uparrow & & & \\ B_c & A_cB_c & A_c^2B_c & A_c^3B_c & \cdots & A_c^{n-1}B_c \end{array}$$

This is upper Δ^{lar} matrix, diagonals are all 1's $\Rightarrow$ rank $= n$

$\Rightarrow (A_c, B_c)$ is controllable

$\Rightarrow (A, B)$ is controllable.

$$\underset{A_c}{TAT^{-1}} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots\; 1 & 0 \end{bmatrix}, \quad \underset{B_c}{TB} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Companion form exists $\Leftrightarrow$ Controllable.

THEOREM

(A, B) is controllable $\overset{\Rightarrow}{\Leftrightarrow}$ there is an invertible transformation T that transforms (A, B) into the companion form,

$$TAT^{-1} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad TB = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proof: ($\Rightarrow$) What is unknown on the right hand side? T

Do we know $\{a_1, a_2, \ldots a_n\}$?

Yes, from the characteristic polynomial of A.

same

check:

$A \longrightarrow TAT^{-1}$

known

unknown

known

$$\begin{tikzcd} (A, B) \arrow[r, "T"] \arrow[d] & (\underbrace{TAT^{-1}}_{A_c}, \underbrace{TB}_{B_c}) \arrow[d] \\ W_c = [B \quad AB \dots A^{n-1}B] \arrow[r, leftrightarrow] & W_{cc} = [B_c \quad A_cB_c \dots A_c^{n-1}B_c] \end{tikzcd}$$

how are these related?

$$\rightarrow TB \quad TAT^{-1}TB \quad \dots (TAT^{-1})^{n-1}TB$$

$$[TB \quad TAB \quad \dots TA^{n-1}B]$$

$$T\underbrace{[B \quad AB \quad \dots A^{n-1}B]}_{W_c}$$

$$TW_c = W_{cc}$$

$$\Rightarrow \quad T = W_{cc}W_c^{-1}$$

Solution for the unknown T.

- W_c^{-1} exists because W_c is square (single input) & $\mathrm{rank}(W_c) = n$

Design Algorithm

Control input using state feedback

- Given design specifications
- Choose dynamics (eigenvalues)
- Is system controllable?

NO

YES

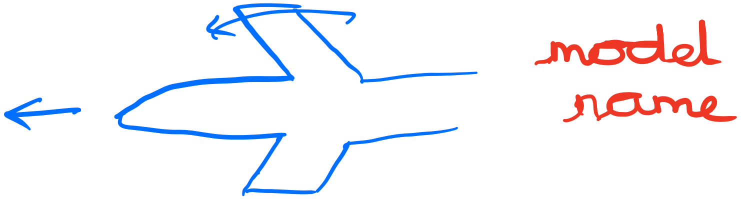
① by hand compute eigenvalues of $A-BK$

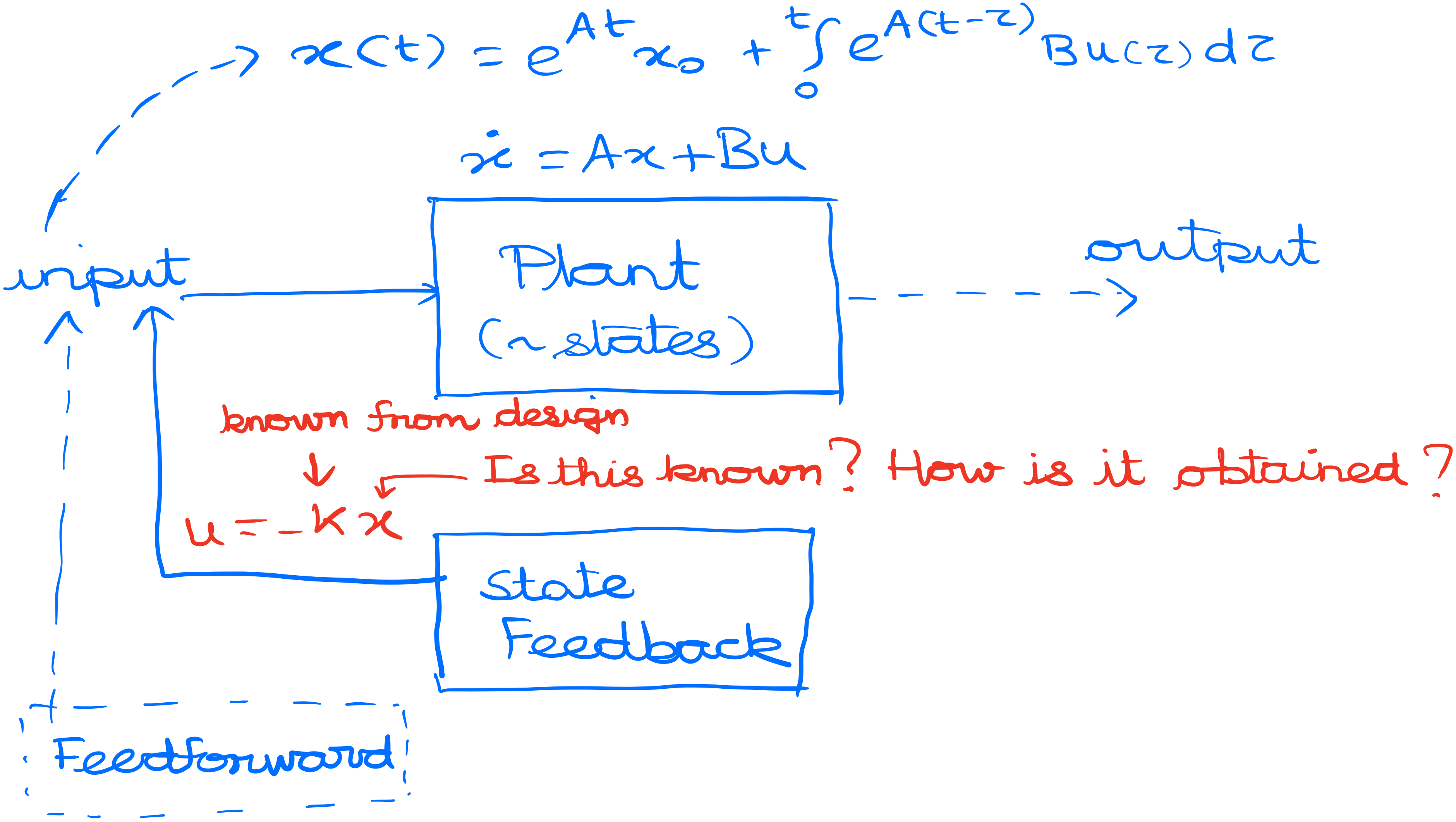
② computationally for larger systems

③ (single input) companion form transformation

uncontrollable parts already satisfy what is desired

some designs intentionally unstable, but controllable




$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$
$\dot{x} = Ax + Bu$
input
Plant
(n states)
output
known from design
Is this known? How is it obtained?
$u = -Kx$
State Feedback
Feedforward