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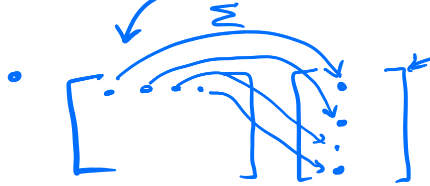
- Is $[TB \;\; TAB \cdots TA^{n-1}B] = T[B \;\; AB \cdots A^{n-1}B]$?

Yes

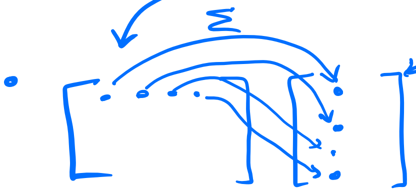
- Take $A_{2\times2}$, $B_{2\times1}$, $T_{2\times2}$ and check

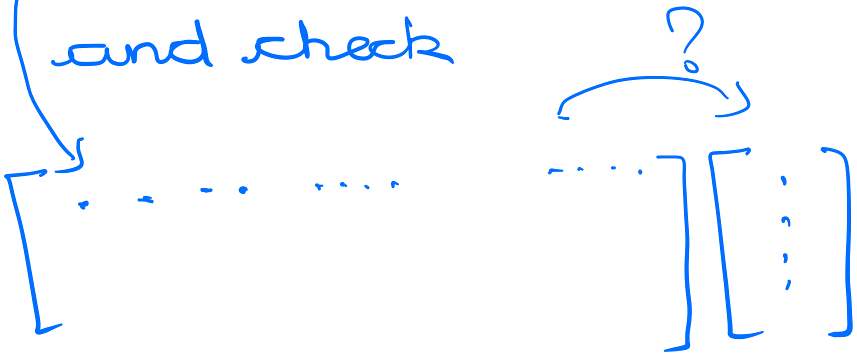


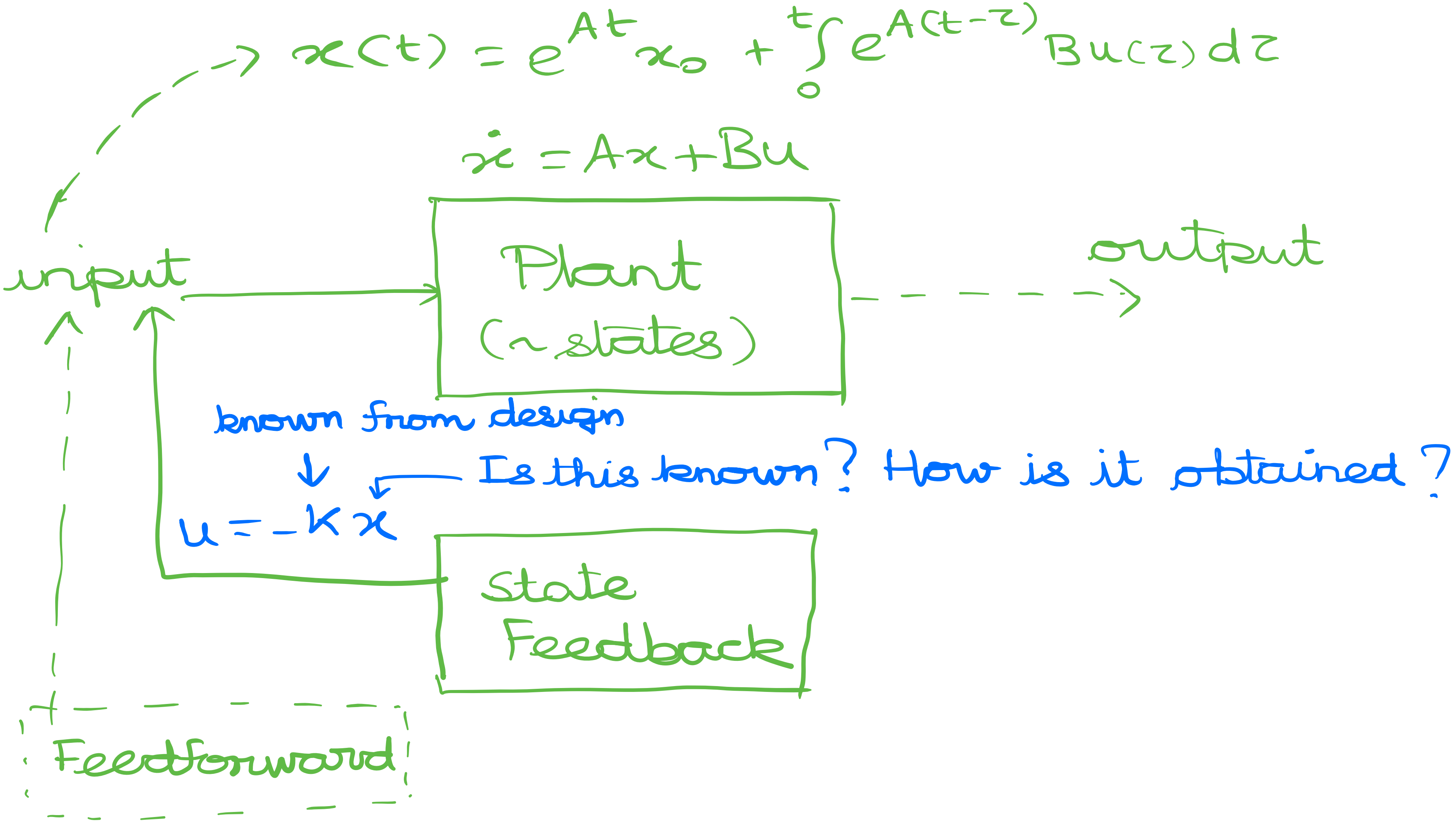
- Is $[TB \;\; TAB \cdots TA^{n-1}B] = [\underbrace{T \;\; TA \cdots TA^{n-1}}_{n \times n^2}]\underset{n \times m}{B}$?

No

- Take $A_{2\times2}$, $B_{2\times1}$, $T_{2\times2}$ and check







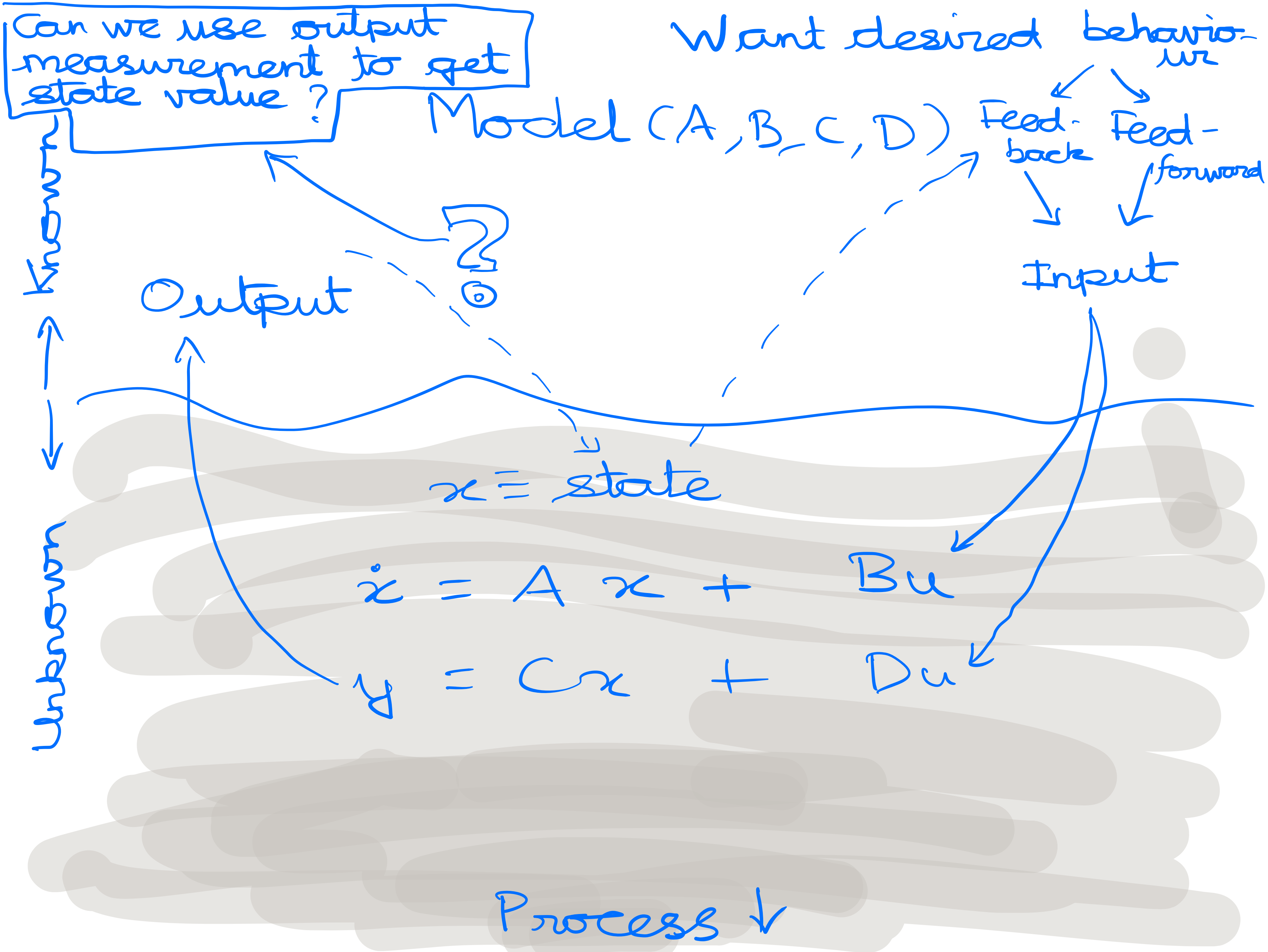
$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

known from design

Is this known? How is it obtained?



Can we use output measurement to get state value ?

Want desired behaviour

Model (A, B, C, D)

x = state

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Process ↓

→ Kalman Filter

Kalman's Example

states: x_1, x_2, x_3, x_4

$y = 7x_1 + 6x_2 + 4x_3 + 2x_4$

input

- $\dot{x}_1 = 2x_1 + 3x_2 + 2x_3 + x_4 + u$

  $\dot{x}_2 = -2x_1 - 3x_2 - 2u$

  $\dot{x}_3 = -2x_1 - 2x_2 - 4x_3 + 2u$

  $\dot{x}_4 = -2x_1 - 2x_2 - 2x_3 - 5x_4 - u$

$$A = \begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix}$$

- $\dot{x} = Ax + Bu$ → $B = [1 \ \ -2 \ \ 2 \ \ -1]'$

→ $y = Cx_{4\times1}$ $(D=0)$

C → $[7 \ \ 6 \ \ 4 \ \ 2]_{1\times4}$

- $C'y = C'Cx$

If $(C'C)^{-1}$ exists, then $x = (C'C)^{-1}C'y$

Check if this is possible...

- Diagonalize A {eigenvalues, eigenvectors}

$$\{\underset{-1}{\lambda_1}, v_1\} \quad \{\underset{-2}{\lambda_2}, v_2\}, \quad \{\underset{-3}{\lambda_3}, v_3\}, \quad \{\underset{-4}{\lambda_4}, v_4\}$$

$$\Rightarrow AV = VD \quad , \quad V = [v_1 \; v_2 \; v_3 \; v_4\}$$

- $z = V^{-1}x \quad , \quad \dot{z} = V^{-1}\dot{x} = V^{-1}Ax + V^{-1}Bu$

$$\Rightarrow \dot{z} = \underbrace{V^{-1}AV}_{D = \text{diag}\{-1,-2,-3,-4\}} z + V^{-1}Bu$$

How does y depend on z?

$$\& \; y = Cx \quad \Rightarrow \quad y = C\underbrace{Vz}_{x}$$

- $CV = [1 \;\; 1 \;\; 0 \;\; 0]$

for this example. Check.

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$\equiv \quad \dot{z}_1 = -z_1 + u$$

$$\dot{z}_2 = -2z_2 \longleftarrow \text{no input}$$

$$\dot{z}_3 = -3z_3 + u$$

$$\dot{z}_4 = -4z_4 \longleftarrow \text{no input}$$

$$y = z_1 + z_2 = \overbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}}^{CV} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

- Can we obtain z_1, z_2, z_3, z_4 from y?

$$((CV)'(CV))^{-1}(CV)'y$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{4\times1} \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}_{1\times4} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ not invertible}$$

Observability

~ The system $\dot{x} = Ax$, $y = Cx$ is observable if it is possible to find any initial state $x(0)$ from the observations of output $y(t)$, $0 \le t \le T$.

Output Solution

- $$\dot{x} = Ax + Bu$$

  $$y = Cx + Du$$

- $$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$\Rightarrow y(t) = \underset{\checkmark}{C}\,\underset{\checkmark}{e^{At}}\,\underset{?}{x(0)} + \underset{\checkmark}{C}\int_0^t \underset{\checkmark}{e^{A(t-\tau)}}\,\underset{\checkmark}{B}\,\underset{\checkmark}{u(\tau)}\, d\tau + \underset{\checkmark}{D}\,\underset{\checkmark}{u}$$

- $$\underbrace{y(t) - C\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau - Du}_{\longrightarrow \text{ considered as output (+ known)}}$$

$$= Ce^{At}x(0)$$

How do we get state from output?

When is this is possible?

$$y(t) = \underbrace{Ce^{At}} x(0)$$

Possible,

When Ce^{At} is "invertible" ~ non singular

~ $\text{rank}\{Ce^{At}\} = n$

~ $\text{nullspace}\{Ce^{At}\} = \emptyset$ (empty)

$= \{ v$: is such that $Ce^{At} v = 0 \}$ $v' Ce^{At} = 0?$

$Ce^{At}(x(0)+v)$

$= Ce^{At}x(0)$

$+ \underbrace{Ce^{At}v}_{=0}$

$v \in \text{Nullspace}\{Ce^{At}\}$

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$$\text{Nullspace}\{Ce^{At}\} \overset{?}{=} \text{Nullspace}\{C\} \cap \text{Nullspace}\{CA\} \cap \cdots\cdots \text{Nullspace}\{CA^{n-1}\}$$

($\subseteq$) $v \in \text{Nullspace}\{Ce^{At}\}$

$\Rightarrow Ce^{At} v = 0$

Set $t=0$ $\Rightarrow$ $Cv = 0$

$\Rightarrow$ $v \in \text{Nullspace}\{C\}$

Differentiate w.r.t 't' $\Rightarrow$ $CAe^{At} v = 0$

& Set $t=0$ $\Rightarrow$ $CAv = 0$ $\Rightarrow$ $v \in \text{Nullspace}\{CA\}$

Continue n-1 times $\Rightarrow$ $CA^{n-1} v = 0$ $\Rightarrow$ $v \in \text{Nullspace}\{CA^{n-1}\}$

$\therefore$ $v \in \text{Nullspace}\{C\} \cap \text{Nullspace}\{CA\} \cap$

($\supseteq$) Suppose $v \in \text{Nullspace}\{C\} \cap \text{Nullspace}\{CA\} \cap$

$\cdots$ $\text{Nullspace}\{CA^{n-1}\}$

$$\Rightarrow Cv = 0 \;\&\; CAv = 0 \;\ldots\; \&\; CA^{n-1}v = 0$$

$$Ce^{At}v = \;?$$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \ldots$$

Scalars

By Cayley-Hamilton Theorem

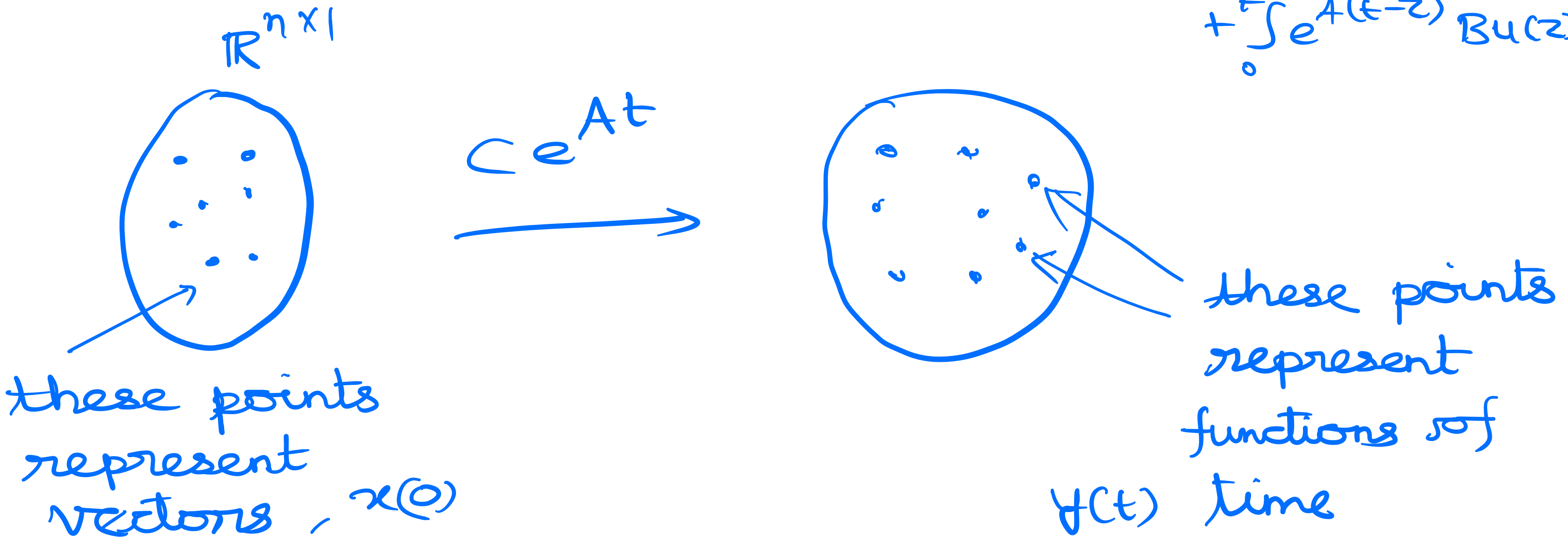
$$e^{At} = f_0(t)\,I + f_1(t)\,A + f_2(t)\,A^2 + \ldots + f_{n-1}(t)\,A^{n-1}$$

$$Ce^{At} = f_0(t)\,C + f_1(t)\,CA + f_2(t)\,CA^2 + \ldots + f_{n-1}(t)\,CA^{n-1}$$

$$Ce^{At}v = f_0(t)\underbrace{Cv}_{\downarrow \atop 0} + f_1(t)\underbrace{CAv}_{\downarrow \atop 0} + \ldots + f_{n-1}(t)\underbrace{CA^{n-1}v}_{\downarrow \atop 0}$$

$$\Rightarrow v \in \text{Nullspace}\{Ce^{At}\}$$

The mapping $y = Ce^{At}x(0)$ is being analysed to see if output measurement ($y(t)$, $0 \leq t \leq T$) can be used to get state information ($x(0) \rightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$)



Claim: If $v (\neq 0) \in \text{Nullspace}\{Ce^{At}\}$, then (A, C) is not observable.

Because any initial state $x(0)$ and the initial state $x(0) + v$ would map to same output

$$x(0) \longrightarrow Ce^{At}x(0)$$

$$x(0)+v \longrightarrow Ce^{At}(x(0)+v)$$

Rewrite

$$= Ce^{At}x(0) + \overset{=0}{\cancel{Ce^{At}v}}$$

$$= Ce^{At}x(0)$$

→ (A,C) is observable $\underset{\overset{?}{\rightarrow}}{\Rightarrow}$ Nullspace$\{Ce^{At}\} = \emptyset$

Check that above proof matches this statement.

(A, C) is observable

$\Rightarrow$ Nullspace$\{Ce^{At}\} = \emptyset$

No $v (\neq 0)$ such that $Cv = CAv = \dots CA^{n-1}v = 0$.

$\Leftrightarrow$ Nullspace$\{C\} \cap$ Nullspace$\{CA\} \cap \dots$

Nullspace$\{CA^{n-1}\} = \emptyset$

$$\Leftrightarrow \quad \text{rank}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = n$$

Combine

$$\underset{p\times n}{C}\ \underset{n\times 1}{v} = 0$$

$$\underset{p\times n}{CA}\ \underset{n\times 1}{v} = 0$$

$$\vdots$$

$$\underset{p\times n}{CA^{n-1}}\ \underset{n\times 1}{v} = 0$$

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np\times n} v_{n\times 1} = 0$$

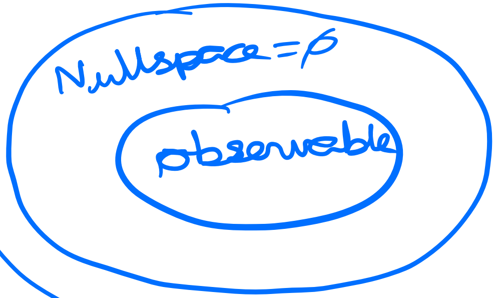
$$\text{Nullspace}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = \emptyset \longleftarrow$$

Rank-Nullity Theorem

(A,C) is observable

$$\Downarrow$$

$$\text{Nullspace}\left\{C e^{At}\right\} = \phi$$



$$\Leftrightarrow \text{Nullspace}\{C\} \cap \text{Nullspace}\{CA\} \ldots \cap \text{Nullspace}\{CA^{n-1}\} = \phi$$

$$\Leftrightarrow \text{Nullspace}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = \phi$$

$$\Leftrightarrow \text{Rank}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = n$$

W_o : Observability matrix

$\text{Nullspace}\{Ce^{At}\} = \phi \Rightarrow (A, C)$ is observable

Suppose (A,C) is not observable

?

$\exists \; v_1, v_2 (\neq v_1)$ such that

$$Ce^{At} v_1 = y(t) = Ce^{At} v_2$$

makes sense

$$\Rightarrow Ce^{At}(v_1 - v_2) = 0$$

$$\Rightarrow \underset{(\neq 0)}{v_1 - v_2} \in \text{Nullspace}\{Ce^{At}\}$$

$\Rightarrow$ Contradiction

$\Rightarrow (A,C)$ is observable.

Nullspace$\{Ce^{At}\} = \phi \Rightarrow (A, C)$ is observable

(a constructive proof . . .)

We need to show

~ The system $\dot{x} = Ax$, $y = Cx$ is observable if it is possible to find any initial state $x(0)$ from the observations of output $y(t)$, $0 \leq t \leq T$.

$$x(0) \xrightarrow{Ce^{At}} y(t)$$

Can we invert this?

$$y(t) = Ce^{At} x(0)$$

$(Ce^{At})' \times [$

$$\Rightarrow (Ce^{At})' y(t) = (Ce^{At})'(Ce^{At}) x(0)$$

$$\Rightarrow \quad e^{A't} C' y(t) = e^{A't} C'C e^{At} x(0)$$

$$\int_0^T \quad \Rightarrow \quad \int_0^T e^{A't} C' y(t) dt = \int_0^T e^{A't} C'C e^{At} dt \, x(0)$$

$$\Rightarrow \quad x(0) = \left(\int_0^T e^{A't} C'C e^{At} dt \right)^{-1} \int_0^T e^{A't} C' y(t) dt$$

if $\underbrace{\left(\int_0^T e^{A't} C'C e^{At} dt \right.}_{\text{Observability Grammian}}\left.\vphantom{\int_0^T}\right)^{-1}$ exists

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if $\left(\underbrace{\int_0^T e^{A't} c' c e^{At} dt}_{\text{Observability Grammian}}\right)^{-1}$ exists

Does this inverse exist ?

Suppose the inverse did not exist

$\Rightarrow$ there is a vector v $(\neq 0)$ such that

$$\int_0^T e^{A't} c' c e^{At} dt \cdot v = 0$$

$v' \times [$ $\Rightarrow$ $v' \left[\int_0^T e^{A't} c' c e^{At} dt\right] v = 0$

$$\Rightarrow \int_0^T [v' e^{A't} c' c e^{At} v] dt = 0$$

$$\Rightarrow \int_0^T \underbrace{[Ce^{At}v]'}_{z'}\underbrace{[Ce^{At}v]}_{z}\, dt = 0$$

—

Suppose $z \in \mathbb{R}$ $\quad \int_0^T z^2\, dt = 0 \Rightarrow z = 0$

$z \in \mathbb{R}^{2\times 1}$ $\quad \int_0^T (z_1^2 + z_2^2)\, dt = 0 \Rightarrow z_1 = 0 = z_2$

—

$$\Rightarrow Ce^{At} v = 0 \quad \& \; v \neq 0$$

contradiction!

We are trying to show

$\text{Nullspace}\{Ce^{At}\} = \phi \Rightarrow (A, C)$ is observable

(A, C) is observable

$\Longleftrightarrow$ Nullspace $\{ Ce^{At} \} = \phi$

$\Longleftrightarrow$ $\text{rank} \left\{ \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \right\} = n$

Bicycle : Observable ?

$$M=\begin{bmatrix}80.8 & 2.3\\ 2.3 & 0.3\end{bmatrix},\ g=9.8,\ v=2$$

$$C_1=\begin{bmatrix}0 & 33.9\\ -0.9 & 1.7\end{bmatrix},\ K_0=\begin{bmatrix}-81 & -2.6\\ -2.6 & -0.8\end{bmatrix}$$

$$K_2=\begin{bmatrix}0 & 76.6\\ 0 & 2.7\end{bmatrix}$$

PARAMETERS

$$\Rightarrow A=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 9.58 & -4.13 & -0.22 & -0.67\\ 12.25 & 21.6 & 7.7 & -6.28\end{bmatrix}$$

- Model

Given
$$\begin{cases} M\ddot{q}+vC_1\dot{q}+\left(v^2K_2+gK_0\right)q=f,\ q=\begin{bmatrix}\phi\\ \delta\end{bmatrix},\ f=\begin{bmatrix}T_\phi\\ T_\delta\end{bmatrix}\\ \phi:\text{ balance angle, } \delta:\text{ steer angle, } f:\text{ input}\end{cases}$$

Set $x_1=\phi,\ x_2=\delta,\ x_3=\dot{\phi},\ x_4=\dot{\delta}$

$$\Rightarrow \underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}}_{\dot{x}} = \underbrace{\left[\begin{array}{cc|cc}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \hline \multicolumn{2}{c|}{-M^{-1}\cdot} & \multicolumn{2}{c}{-M^{-1}vC_1}\\ \multicolumn{2}{c|}{[gK_0+v^2K_2]} & & \end{array}\right]}_{A} \underbrace{\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}}_{x} + \underbrace{\left[\begin{array}{cc}0 & 0\\ 0 & 0\\ \hline \multicolumn{2}{c}{M^{-1}}\end{array}\right]}_{B} \underbrace{\begin{bmatrix}T_\phi\\ T_\delta\end{bmatrix}}_{u}$$

$$\dot{x} = A\,x + B\,u$$

$$\text{rank}\begin{bmatrix}C\\ CA\\ CA^2\\ CA^3\end{bmatrix}=4\ ?$$

- Is the system observable if only output is a) ϕ, b) δ, c) $\dot{\phi}$, d) $\phi-\delta$.

$C=[1\ 0\ 0\ 0]$ $[0\ 1\ 0\ 0]$ $[0\ 0\ 1\ 0]$ $[1\ -1\ 0\ 0]$

(a) $c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$

$$\begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ a_1 a_3 + b_1 a_4 & a_2 a_3 + b_2 a_4 & a_3 a_3 + b_3 a_4 & a_4 a_3 + b_4 b_4 \end{bmatrix}$$

- ~ dynamics are coupled, so these "limited" measurements suffice for observability

- Duality between observability and controllability

$$\text{rank}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = n \qquad \text{rank}\{[B \ \ AB \ \cdots \ A^{n-1}B]\} = n$$

- Theorem:

(A, C) is observable

not same system

$\Leftrightarrow$ (A', C') is controllable.

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}' = [C' \ (CA)' \ \cdots \ (CA^{n-1})']$$

$$= [C' \ \ A'C' \ \cdots \ (A')^{n-1}C']$$

Duality can be used to prove...

- Observability is independent of co-ordinate transformation

- Suppose $\text{rank}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = r < n$

  there exists a co-ordinate transformation such that...

+ How to design observers?

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Minor Exam

- 28.09.2022 , Wednesday

  8 -9 AM

  LH 408

- Calculators

~ Observability

- observe states from output if

$$\text{rank}\left\{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right\} = n$$

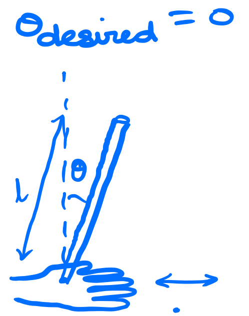
$\downarrow$

- how to design observers?

↳ input : measurement + model of dynamics

↳ Should we use this? A, C

$$x(0) = \left(\int_0^T e^{A't} C' C e^{At} dt\right)^{-1} \int_0^T e^{A't} C' y(t) dt$$



states = θ, $\dot{\theta}$

control, $u = -k\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$

observable?

observer design?

Maybe this is how observers were designed ?

x → state (fn. of time) ,

'hat'

state

$\hat{x}$ → estimate

e → error, $e = x - \hat{x}$

We want the error to be zero.

We are observing

$$\dot{x} = Ax$$

$$y = Cx$$

We are measuring

unknown

$$\dot{\hat{x}} = \hat{A}\hat{x} + Ly$$

$$\hat{y} = \hat{C}\hat{x}$$

For $\dot{\hat{x}} = \hat{A}\hat{x} + Ly$

$$= \hat{A}\hat{x} + LCx$$

$e = x - \hat{x}$ follows what equation?

$$\dot{e} = \dot{x} - \dot{\hat{x}}$$

$$= Ax - \hat{A}\hat{x} - LCx$$

$$= (A-LC)x - \hat{A}(\overbrace{x-e}^{\hat{x}})$$

$$\Rightarrow \dot{e} = \hat{A}e + (A-LC-\hat{A})x$$

Choose $\hat{A} = A - LC$

$$\Rightarrow \dot{e} = (A-LC)e$$

Can we choose L so that $e(t) \to 0$?

For $e(t) \to 0$,

- eigenvalues of $A-LC$ should have negative real part.

Do we know that the eigenvalues of $A-LC$ can be placed anywhere by suitable choice of L?

↪ Is it always possible?

↪ What are the conditions for it?

In control design, we had a similar question: Can the eigenvalues of $A-BK$ be placed anywhere by a proper choice of K?

- Yes, if (A,B) are controllable.

Invoke the duality between controllability and observability to solve the eigenvalue assignment problem ...

- To place eigenvalues of $A-LC$
- we can place eigenvalues of $(A-LC)'$ $= A' - C'L'$
- this is always possible if

(A', C') is controllable. $\rightarrow \text{rank}[C' \; A'C' \ldots (A')^{n-1} C'] = n$

- which means possible if (A, C) is observable.

take Transpose

$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

System $\longrightarrow$ Observer

$$\dot{x} = Ax$$
$$y = Cx$$

$$\dot{\hat{x}} = \underset{\overset{\uparrow}{A-LC}}{\hat{A}} \hat{x} + Ly$$

$$\dot{\hat{x}} = (A - LC) \hat{x} + Ly$$

$$\Rightarrow \dot{\hat{x}} = A\hat{x} + (-LC)\hat{x} + Ly$$

$$= A\hat{x} + L(y - \underbrace{C\hat{x}}_{\hat{y}})$$