10.10.2022

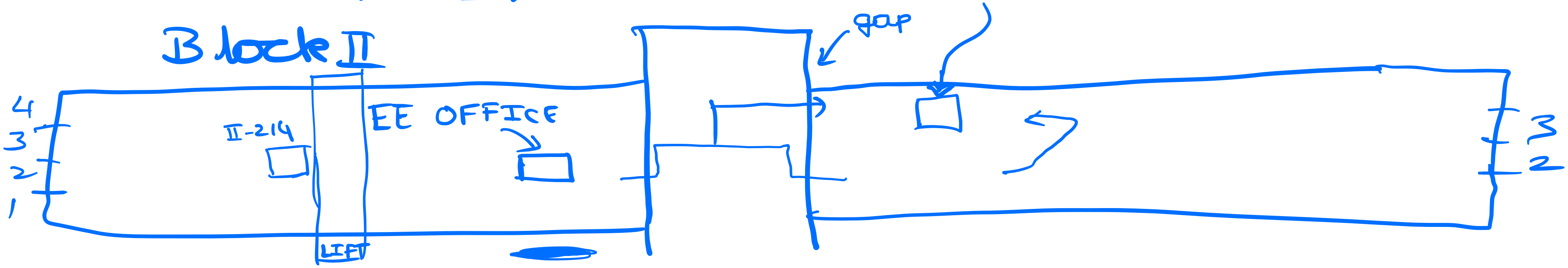
- Check calculation of 1D example design

  https://web.iitd.ac.in/~shaunak/sen/2022Sem1_ELL333/ELL333_Design_1Dexample.jl.html

- Project report → submit in next class (LH 316)

- Minor Exam → II-339 B

Block II



$\odot$

- $\dot{x} = Ax + Bu$

$n \times 1$ $m \times 1$

$p \times 1$

$y = Cx + Du$

- Transfer Function

Laplace Transform of

$sX(s) - \cancel{x(0)} = AX(s) + BU(s)$

I $=0$

$(sI - A)X(s) = BU(s)$

$X(s) = (sI-A)^{-1}BU(s)$

Transfer Function

$Y(s) = (C(sI-A)^{-1}B + D)U(s)$

⊙ Transfer Function

$z = Tx$
$\dot{z} = T\dot{x}$
$= TAx + TBu$
$= TAT^{-1}z + TBu$
$y = CT^{-1}z + Du$

$$C(sI-A)^{-1}B + D$$

- If $(A, B, C, D) \xrightarrow[\text{similarity}]{} (TAT^{-1}, TB, CT^{-1}, D)$,

then transfer function is ?

$$CT^{-1}(\underset{\overset{\uparrow}{TT^{-1}}}{sI} - TAT^{-1})^{-1}TB + D$$

$$= CT^{-1}(T(sI-A)T^{-1})^{-1}TB + D$$

$$= \underbrace{CT^{-1}(T^{-1})^{-1}}_{I}(sI-A)^{-1}\underbrace{T^{-1}T}_{I}B + D$$

= same, as the relationship between u & y unchanged.

* Transfer Function $= \dfrac{N(s)}{D(s)}$ matrix

$$(sI-A)^{-1} = \frac{\text{adj}\{(sI-A)\}}{\det\{(sI-A)\}}$$

Roots of $D(s) = \det\{(sI-A)\}$

are poles $=$ eigenvalues of A.

Calculate transfer function

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}_{4\times4}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}_{4\times1}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}_{1\times4}, \quad D = 0$$

$$C(sI-A)^{-1}B = \frac{1}{(s+2)(s+3)(s+4)}$$

$$\frac{1}{s+1} \quad ? \quad \frac{(s+2)(s+3)(s+4)}{(s+1)(s+2)(s+3)(s+4)}$$

Pole-Zero cancellation

13.10.2022

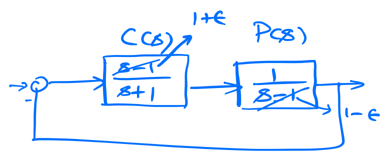
$$\underset{1\times 4}{C}\;\underset{4\times 4}{(sI-A)^{-1}}\;\underset{4\times 1}{B} = \frac{1}{(s+2)(s+3)(s+4)}$$

✓ $$\frac{1}{s+1}$$?

$$\frac{\cancel{(s+2)}\cancel{(s+3)}\cancel{(s+4)}}{(s+1)\cancel{(s+2)}\cancel{(s+3)}\cancel{(s+4)}}$$ ✓

Pole-Zero cancellation

→ Pole-Zero Cancellation

- indicates loss of controllability / observa-bility
- practically 

, exact cancellation not possible ⇒ responses may "blow up".

$$\underset{p\times 1}{Y(s)} = G(s)\, U(s)_{m\times 1}$$

$$\underset{p\times m}{G(s)} = C(sI-A)^{-1}B + D$$

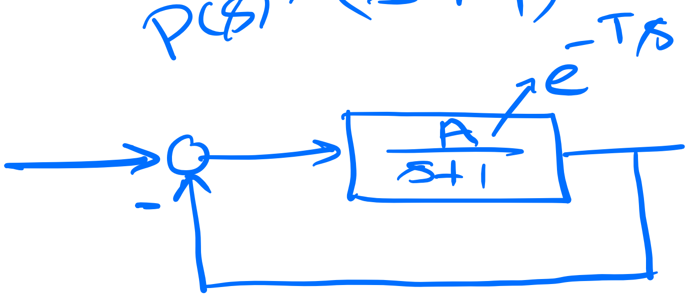
$$\begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2m}(s) \\ \cdots & \cdots & \cdots & \cdots \\ G_{p1}(s) & G_{p2}(s) & \cdots & G_{pm}(s) \end{bmatrix}_{p\times m}$$

pm transfer functions need to be visualised.

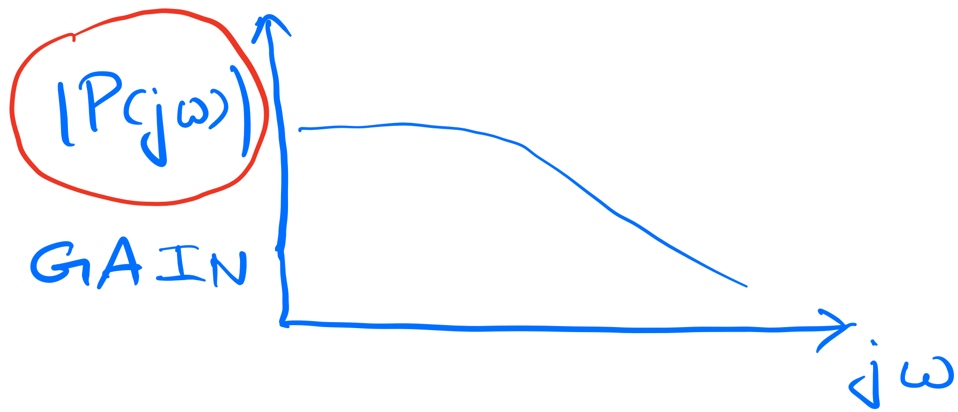
What do we associate with a transfer function in design of Single Input Single Output (SISO) designs (ELL225)?

Example: $P(s) = \frac{A}{(s+1)} \cdot e^{-Ts}$ ... Bode plots

Nyquist plots



$$s + 1 + A e^{-Ts} = 0$$



Gain for multivariable system's frequency response

- gain of $\underset{p\times m}{G(s)}$?
- gain/size of a vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$? = norms
- gain/size of a matrix A ? = induced norm (singular values)

(need not be square)

gain/size of a vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$?

= norm/distance

✓ $\to \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ Euclidean 2-norm

$\to |x_1| + |x_2| + \cdots + |x_n|$ 1-norm

$\to \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$ p-norm

✓ $\to \max\{|x_1|, |x_2|, \ldots |x_n|\}$ ∞-norm gain / size of a matrix A (need not be square)

$$A_{p \times m} \text{ linear map}$$

$$\underset{(u)}{\mathbb{R}^m} \longrightarrow \underset{(y = Au)}{\mathbb{R}^p}$$

= induced matrix norm, norm induced by the vector norms

$$\|A\|_2 = \max_u \frac{\|y\|_2}{\|u\|_2}$$ (Is this a norm?)

as $\|u\|_2^2 = u'u$

$$\|y\|_2^2 = y'y = (Au)'(Au) = u'A'Au$$

$$\Rightarrow \quad \|A\|_2 = \max_u \sqrt{\frac{u'A'Au}{u'u}}$$

$$\underbrace{u' A' A}_{M} \; u$$

M → symmetric matrix

$m \times m$ square

eigenvalues: $\underbrace{\lambda_1, \lambda_2, \ldots \lambda_m}$ → denote the maximum by λ_{max}

[$\triangleq$ singular values of A]

In the eigenvector direction, $\bar{u}$

corresponding to λ_{max},

$$\|A\|_2 = \max_{u} \sqrt{\frac{\bar{u}' M \bar{u}}{\bar{u}' \bar{u}}} = \sqrt{\frac{\bar{u}' \lambda_{max} \bar{u}}{\bar{u}' \bar{u}}}$$

$= \bar{u}$

$$= \sqrt{\lambda_{max}} \sqrt{\frac{\bar{u}' \bar{u}}{\bar{u}' \bar{u}}} = \sqrt{\lambda_{max}}$$

Induced matrix norm $\|A\|_2 = \sqrt{\lambda_{max}(A'A)}$

is this positive

$$u'A'Au = (uA)'(Au)$$
$$= \|Au\|_2^2$$
$$\geq 0$$

$\underbrace{u' A' A}_{M} u$: Quadratic Form

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$$

$$v'v = x^2 + y^2$$

$$v' M v = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

M ↓ symmetric

$$\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \quad = \begin{bmatrix} ax + \frac{b}{2}y & \frac{b}{2}x + cy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$Q(x,y) = ax^2 + bxy + cy^2$$

For a fixed M on $\{a, b, c\}$,

is the sign of $Q(x,y)$ definitely known?

ellipse	$Q(x,y) > 0$		positive definite
parabolas	$Q(x,y) \geq 0$	$\forall\, x, y \in \mathbb{R}^2 - \{0\}$	positive semi definite
parabolas	$Q(x,y) < 0$		negative definite
ellipse	$Q(x,y) \leq 0$		negative semi definite
hyperbola	$Q(x,y) \gtrless 0$	$\longrightarrow$	indefinite

17.10.2022

- Transfer Function matrix → Gain ?

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① static $y = Ax$

- "size" of a vector

  ~ norm, ex: $\|\cdot\|_2$

- "size" of a matrix

  ~ induced norm

$$\Rightarrow \|A\|_2 \triangleq \max_x \frac{\|Ax\|_2}{\|x\|_2}$$

dynamic map

- $$y(t) = \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$\downarrow \text{map}$$

$$u(t)_{m\times 1} \longrightarrow y(t)_{p\times 1}$$

Vector space not $\mathbb{R}^n$

but $L_2[0,\infty) = \left\{ u(t) : \int_0^\infty |u(t)|^2\, dt < \infty \right\}$

$$L_2\,\text{norm}(u) = \int_0^\infty |u(t)|^2\, dt$$

. . . induced norms have to be defined in this space . . .

- $$\int_0^\infty |y(t)|^2 \, dt = (\quad) \int_0^\infty |Y(j\omega)|^2 \, d\omega$$

$$= (\quad) \int_0^\infty |G(j\omega) U(j\omega)|^2 \, d\omega$$

$$= (\quad) \int_0^\infty |G(j\omega)|^2 \, |U(j\omega)|^2 \, d\omega$$

If G was scalar frequency response

$$\leq \max_\omega |G(j\omega)|^2 (\quad) \int_0^\infty |U(j\omega)|^2 \, d\omega$$

$$\leq \max_\omega |G(j\omega)|^2 \int_0^\infty |u(t)|^2 \, dt$$

If G was vector $\rightarrow \|G\|_\infty$

→ Robust Control

→ Optimal Control

→ Stochastic Filtering & Identification

→ ~ Robotics

→ ~ Adaptive Control

→ Special Topics

Discrete Models

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

$x(t)$

t : continuous

above below

$A \neq A$

↓ same notation, but different meaning

$$x_{k+1} = A x_k + B u_k$$

$$y_k = C x_k + D u_k$$

$x[k]$

$\equiv x_k$

Why discrete? → ease of computing

→ Σ instead of $\int$

Stability

$\dot{x} = Ax$, eig$\{A\} \in$ LHP
$\Rightarrow$ Stability

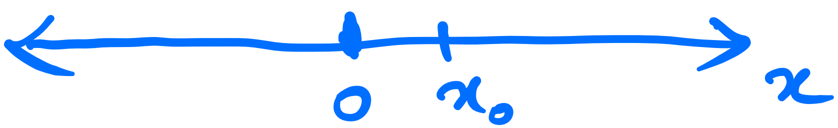
$$x_{k+1} = A x_k$$

$$= A \cdot A x_{k-1}$$

$$= \underbrace{A \cdot A \cdots A}_{(k+1) \text{ times}} \cdot x_0$$

$$= A^{k+1} x_0$$

Stability at $x = 0$



What is condition on A so that $x_k \to 0$?