Discrete Models

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

$x(t)$

t : continuous

above below

$A \neq A$

↓ same notation, but different meaning

$$x_{k+1} = A x_k + B u_k \qquad x[k]$$

$$y_k = C x_k + D u_k \qquad \equiv x_k$$

Why discrete? → ease of computing

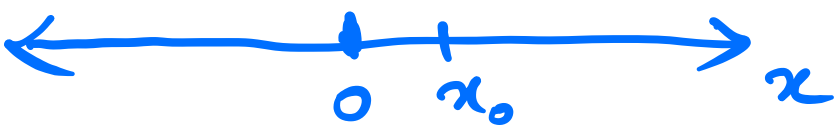
→ Σ instead of $\int$

Stability

$\dot{x} = Ax$, eig$\{A\} \in$ LHP
$\Rightarrow$ Stability

$$
\begin{aligned}
x_{k+1} &= A x_k \\
&= A \cdot A x_{k-1} \\
&= \underbrace{A \cdot A \cdots A}_{(k+1)\text{ times}} \cdot x_0 \\
&= A^{k+1} x_0
\end{aligned}
$$

Stability at $x = 0$



What is condition on A so that $x_k \to 0$?

Change co-ordinate frame ...

$$x_{k+1} = A \, x_k$$

- Diagonalise A ≡ Find T such that

$$T^{-1} A T = D \quad \text{(diagonal)}$$

- $z_k = T^{-1} x_k$ is co-ordinate trans-formation

$$\Rightarrow z_{k+1} = T^{-1} x_{k+1}$$

$$= T^{-1} A x_k$$

$$= T^{-1} A T z_k$$

$$= D z_k$$

{same as in continuous time}

$$z_{i,k+1} = \lambda_i z_{i,k}$$

$\mathrm{diag}\{\lambda_1, \lambda_2, .. \lambda_n\}$

$$\begin{bmatrix} z_{1k} \\ z_{2k} \\ \vdots \\ z_{nk} \end{bmatrix}$$

as $\{x_{1k}, x_{2k}, \dots x_{nk}\}$ are just linear combinations of $\{z_{1k}, z_{2k}, \dots z_{nk}\}$, via the co-ordinate transformation T,

if any $|\lambda_i| > 1$,

corresponding $z_{ik} \to \infty$ as $k \to \infty$

$\Rightarrow$ one or more $x_{ik} \to \infty$ as $k \to \infty$

$u_k = -K x_k$

~ Controllability

$$x_{k+1} = A x_k + B u_k, \quad x_0 = 0$$

$$x_1 = B u_0$$

$$x_2 = A x_1 + B u_1$$

$$= AB u_0 + B u_1$$

$$x_3 = A x_2 + B u_2$$

$$= \underline{A^2 B} u_0 + \underline{AB} u_1 + \underline{B} u_2$$

$$x_4 = A x_3 + B u_3$$

$$= A^3 B u_0 + A^2 B u_1 + AB u_2 + B u_3$$

$$x_n = A^{n-1}Bu_0 + A^{n-2}Bu_1 + \dots + ABu_{n-2} + Bu_{n-1}$$

$$x_n = \underbrace{[\underbrace{B}_{n\times m}\quad \underbrace{AB}_{n\times m}\quad A^2B\quad \dots\quad A^{n-1}B]}_{n\times nm}\ \underbrace{\begin{array}{cc}\begin{bmatrix}u_{n-1}\\ u_{n-2}\\ \vdots\\ u_0\end{bmatrix} & \begin{array}{c} m\times 1\\ m\times 1\\ \\ m\times 1\end{array}\end{array}}_{(nm\times 1)}$$

If $[B\ AB\ A^2B \dots A^{n-1}B]$ had rank $< n$, then we would not be able to reach some $x_n \Rightarrow$ not controllable.

~ Observability

$$x_{k+1} = A x_k$$

for here ($B=0$)

$$y_k = C x_k$$

Given $\{y_k\}_{k=0}^{?}$, what is x_0?

$$y_0 = C x_0$$

Is C invertible?
Suppose $C=[1\,0\,0 \cdots 0]$
Can I do $(C^TC)^{-1}C^T$
but $C^TC = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & & & \\ 0 & \cdots & & & 0 \end{bmatrix}$

$$y_1 = Cx_1 = CAx_0$$

$$\vdots$$

$$y_{n-1} = \cdots = CA^{n-1}x_0$$

Why stop here?
$y_n, y_{n+1} \ldots$
Cayley-Ham

matrix equation?

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np \times n} x_0$$

If $\text{nullspace}\left\{ \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \right\} \neq \phi$, then

there would be some x_0 that

cannot be estimated.