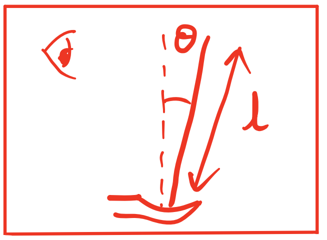
ELL333 Major Exam 2 hr, {34 marks}

Solutions

A model of an inverted pendulum (length: l) is

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ \frac{2l}{1+l^2}\theta + \frac{1}{2}\underbrace{u}_{\text{input}}\end{bmatrix}$$



output, $y = \theta$

It is desired to balance this in the inverted position.

1. Using $x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$, represent the model in the state-space form, $\dot{x} = Ax + Bu$, $y = Cx + Du$.
Show that $A = \begin{bmatrix}0 & 1 \\ \frac{2l}{1+l^2} & 0\end{bmatrix}$, $B = \begin{bmatrix}0 \\ 1/2\end{bmatrix}$, $C = \begin{bmatrix}1 & 0\end{bmatrix}$, $D = 0$. {2}

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 \\ \frac{2l}{1+l^2} & 0\end{bmatrix}}_{A}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} + \underbrace{\begin{bmatrix}0 \\ 1/2\end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} + \underbrace{0}_{D} . u$$

2. Analyse the stability of the system {2}

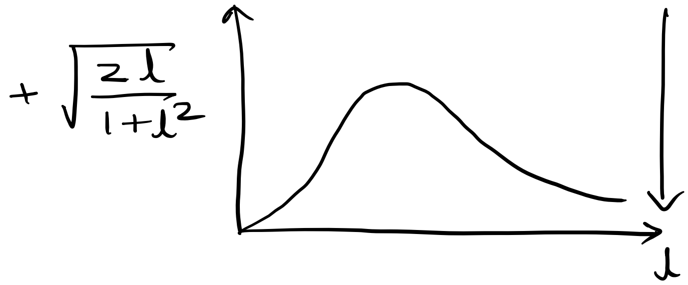
Eigenvalues of A are $\pm\sqrt{\frac{2l}{1+l^2}}$.

One is positive

$\Rightarrow$ Unstable.

3. Based on the model or otherwise, discuss how the ease of balancing depends on the length l. {2}

The ease of balancing would depend on how small the unstable eigenvalue is.



easier (based on this model)

For questions below, set $l=1$.

4. What are the eigenvectors of A? {3}

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, eigenvalues $= \pm 1$

For $\lambda = +1$, $\lambda I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ Eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = -1$, $\lambda I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ Eigenvector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

5. Calculate e^{At}. {3}

$$A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1},$$

$$= \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix} \cdot \frac{1}{(-2)} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix},$$

$$= \frac{1}{2} \begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix}.$$

6. Show that $\frac{d}{dt}\det(e^{At}) = \operatorname{trace}(A).\det(e^{At})$ {3}

$$\text{RHS} = \operatorname{trace}(A).\det(e^{At}) = 0$$

$$\det(e^{At}) = \frac{1}{2}\left[(e^{t}+e^{-t})^2 - (e^{t}-e^{-t})^2\right]$$

$$= \frac{1}{2}\, 2e^{t} \cdot 2 \cdot e^{-t}$$

$$= 2$$

$$\Rightarrow \text{LHS} = 0.$$

7. Show that the system is controllable. {2}

$$\operatorname{rank}\{[B \;\; AB]\} = \operatorname{rank}\left\{\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}\right\} = 2$$

$\Rightarrow$ controllable.

8. Design a state feedback control $u = -Kx$ so that the eigenvalues of the closed loop system are at the roots of the quadratic equation $s^2 + \frac{5}{2}s + \frac{3}{2} = 0$. {3}

$u = -Kx$

$\Rightarrow \dot{x} = (A - BK)x$

$$A - BK = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 - \frac{k_1}{2} & -\frac{k_2}{2} \end{bmatrix}$$

Characteristic Polynomial is

$$s\left(s + \frac{k_2}{2}\right) - \left(1 - \frac{k_1}{2}\right) = 0$$

$$\Rightarrow s^2 + \frac{k_2}{2}s + \left(\frac{k_1}{2} - 1\right) = 0$$

By inspection, $k_2 = 5$, $k_1 = 5$.

9. Calculate the transfer functions,

(a) $C(sI - A)^{-1}$,

(b) $C(sI - (A - BK))^{-1}$. {3}

(a) $$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix}}{s^2 - 1} \cdot \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}$$

$$= \frac{\begin{bmatrix} s & 1 \end{bmatrix}}{s^2 - 1}$$

(b) $[1 \quad 0]\begin{bmatrix} s & -1 \\ \frac{3}{2} & s+\frac{5}{2}\end{bmatrix}^{-1} = \dfrac{[1 \quad 0]}{s^2+\frac{5}{2}s+\frac{3}{2}}\begin{bmatrix} s+\frac{5}{2} & 1 \\ -\frac{3}{2} & s\end{bmatrix}$

$$= \frac{[\,s+\frac{5}{2} \quad 1\,]}{s^2+\frac{5}{2}s+\frac{3}{2}}$$

10. Show that the system is observable. {2}

$$\text{rank}\left\{\begin{bmatrix} C \\ CA \end{bmatrix}\right\} = \text{rank}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = 2$$

$\Rightarrow$ Observable.

11. Design an observer (assume $u=0$)

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly$$

so that the eigenvalues of $A-LC$ are at the roots of the quadratic equation $s^2+2s+2=0$. {3}

$$A-LC = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}[1 \quad 0]$$

$$= \begin{bmatrix} -l_1 & 1 \\ 1-l_2 & 0 \end{bmatrix}$$

$\Rightarrow$ Characteristic Polynomial

$$(s+l_1)s - (1-l_2) = 0$$

$$\Rightarrow \quad s^2 + l_1 s - (1-l_2) = 0$$

By inspection $l_1 = 2$, $l_2 = 3$.

12. Write the equations of the overall controller - observer. {2}

$$\left.\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}\right\} \text{Plant}$$

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} u + \begin{bmatrix} 2 \\ -1 \end{bmatrix} (y - [1 \ 0]x)$$

$$u = -[5 \ \ 5] \hat{x}$$

overall controller - observer.

The Linear Quadratic Regulator (LQR) that minimises the cost $J = \int_0^\infty (x^T Q x + u^T R u)\,dt$ is $u = -R^{-1} B^T P x$ where P is a symmetric matrix that satisfies the equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

13. For $Q = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$, $R = 1$, find the LQR gain $K_{lqr} = R^{-1} B^T P$. {4}

(Hint: Eigenvalues of $A - BK_{lqr}$ have negative real part)

Set $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ (symmetric)

$$A^T P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{12} & p_{22} \\ p_{11} & p_{12} \end{bmatrix}$$

$$P A = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_{12} & p_{11} \\ p_{22} & p_{12} \end{bmatrix}$$

$$P B R^{-1} B^T P = P \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \end{bmatrix} P$$

$$= P \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{p_{12}}{4} & \frac{p_{22}}{4} \end{bmatrix}$$

$$= \begin{bmatrix} p_{12}^2/4 & p_{12} p_{22}/4 \\ p_{12} p_{22}/4 & p_{22}^2/4 \end{bmatrix}$$

$$\therefore \quad 2 p_{12} - p_{12}^2/4 + 5 = 0$$

$$p_{22} + p_{11} - p_{12} p_{22}/4 = 0$$

$$2 p_{12} - p_{22}^2/4 + 5 = 0$$

$$\Rightarrow \quad p_{12} = \frac{-2 \pm \sqrt{4 + 5}}{(-1/2)} = 10, -2$$

$$\Rightarrow \quad p_{22} = \pm 2 \{5, 1\} = \pm 10, \pm 2$$

$$K_{lqr} = R^{-1} B^T P$$

$$= \begin{bmatrix} 0 & 1/2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$= \begin{bmatrix} p_{12}/2 & p_{22}/2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \end{bmatrix}.$$

Check that other choices do not give $\mathrm{Re}\{\mathrm{eig}(A - BK)\} < 0$.