ELL333 Minor Exam 1 hr, 17 marks.

Solutions

{4 marks}

1. For a 2 × 2 symmetric matrix,

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad a, b, c \in \mathbb{R}$$

(a) calculate eigenvalues and their corresponding eigenvectors.

(b) what are the conditions on $\{a, b, c\}$ for

(i) the eigenvalues to be real?

(ii) the eigenvectors to be orthogonal?

a solution:

(a) For eigenvalues, calculate $|\lambda I - A| = 0$

$$\Rightarrow \begin{vmatrix} \lambda - a & -b \\ -b & \lambda - c \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - (a+c)\lambda + (ac - b^2) = 0$$

$$\Rightarrow \quad \lambda_\pm = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac - b^2)}}{2}$$

$$= \frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}$$

1

Note how these are always real as

the term under the square root is positive. ← (b. i) 1

For eigenvalues, calculate nullspace of $\lambda I - A$ for $\lambda = \lambda_\pm$.

$$\lambda_\pm I - A = \begin{bmatrix} \lambda_\pm - a & -b \\ -b & \lambda_\pm - c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -b/(\lambda_\pm - a) \\ -b & \lambda_\pm - c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -b/(\lambda_\pm - a) \\ 0 & 0 \end{bmatrix}$$

because

$$-\frac{b^2}{(\lambda_\pm - a)} + (\lambda_\pm - c) = \frac{-b^2 + (\lambda_\pm - a)(\lambda_\pm - c)}{(\lambda_\pm - a)} = 0$$

Therefore, eigenvectors are $\begin{bmatrix} v_\pm \\ 1 \end{bmatrix}$,

where $v_\pm = \dfrac{b}{\lambda_\pm - a}$. 1

Note that $[v_+ \;\; 1]' \begin{bmatrix} v_- \\ 1 \end{bmatrix} = v_+ v_- + 1$

$$= \frac{b}{(\lambda_+ - a)(\lambda_- - a)} + 1$$

$$= \frac{b + \lambda_+\lambda_- - a(\lambda_+ + \lambda_-) + a^2}{(\lambda_+ - a)(\lambda_- - a)}$$

$$= \frac{b + \overbrace{(ac - b^2)}^{\text{product of roots}} - a \cdot \overbrace{(a+c)}^{\text{sum of roots}} + a^2}{(\lambda_+ - a)(\lambda_- - a)}$$

$$= 0$$

(b.ii)

$\Rightarrow$ Eigenvectors are orthogonal. 1

For symmetric matrices, eigenvalues are real and eigenvectors are orthogonal.

When are the eigenvalues positive?

{5 marks}

2. Show that

(A, B) is controllable $\iff$

$$\text{rank}\,[A - \lambda I \;\; B] = n,$$

for all complex numbers λ.

($\Rightarrow$) Suppose $\text{rank}\,[A - \lambda I \;\; B] < n$ for a complex number λ.

$\Rightarrow \exists\; v \neq 0$ such that

$$v' [A - \lambda I \;\; B] = 0$$

$$\Rightarrow [v'(A - \lambda I) \;\; v'B] = 0$$

$$\Rightarrow v'(A - \lambda I) = 0, \; v'B = 0$$

$$\Rightarrow v'A = \lambda v', \; v'B = 0$$ 1

Consider $v' [B \;\; AB \; \dots \; A^{n-1} B]$

$$= [v'B \;\; v'AB \dots \; v'A^{n-1} B]$$

$$= [0 \;\; \lambda v'B \; \dots \; \lambda^{n-1} v'B]$$

$$= [0 \quad 0 \quad \ldots \quad 0]$$

$$\Rightarrow \quad \text{rank}\{[B \quad AB \quad \ldots \quad A^{n-1}B]\} < n$$

$\Rightarrow$ (A, B) is not controllable.

This is a contradiction!

($\Leftarrow$) Suppose (A, B) is not controllable

$$\Rightarrow \quad \text{rank}\{[B \quad AB \quad \ldots \quad A^{n-1}B]\} < n$$

Let $\text{rank}\{[B \;\; AB \ldots \; A^{n-1}B]\} = r < n$

Then there exists a state transformation T such that

(i) $$T^{-1}AT = \overset{\xrightarrow{\;\;} \tilde{A}}{\left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline 0 & \tilde{A}_{22} \end{array}\right]} \begin{array}{l} \}r \\ \}n-r \end{array} \qquad \begin{array}{cc} \underbrace{\hphantom{\tilde{A}_{11}}}_{r} & \underbrace{\hphantom{\tilde{A}_{22}}}_{n-r} \end{array} \quad , \quad T^{-1}B = \overset{\xrightarrow{\;\;} \tilde{B}}{\left[\begin{array}{c} \tilde{B}_1 \\ \hline 0 \end{array}\right]} \begin{array}{l} \}r \\ \}n-r \end{array}$$

(ii) $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable.

$$\begin{bmatrix} \lambda I - A & B \end{bmatrix}$$

$$= \begin{bmatrix} \lambda I - T\tilde{A}T^{-1} & T\tilde{B} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda T T^{-1} - T\tilde{A}T^{-1} & T\tilde{B} \end{bmatrix}_{n \times (n+m)}$$

$$= T \begin{bmatrix} \lambda I - \tilde{A} & \tilde{B} \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}_{(n+m) \times (n+m)}$$

$$\Rightarrow \text{rank}\{[\lambda I - A \;\; B]\} = \text{rank}\{[\lambda I - \hat{A} \;\; \tilde{B}]\}$$

But $\begin{bmatrix} \lambda I - \tilde{A} & \tilde{B} \end{bmatrix}$

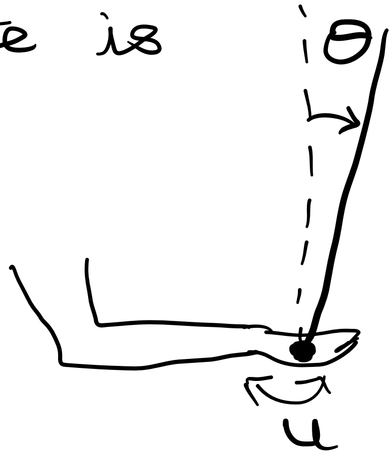
$$= \begin{bmatrix} \lambda I - \tilde{A}_{11} & -\tilde{A}_{12} & \tilde{B}_1 \\ 0 & \lambda I - \tilde{A}_{22} & 0 \end{bmatrix}$$

loses rank at the eigenvalues of $\tilde{A}_{22}$.

This is a contradiction!

{5 marks}

3. A simplified model of "balancing a stick on the hand", when the hand does not transtate is



$$\dot{\theta} = \omega$$
$$\dot{\omega} = \Omega^2 \theta - \alpha\omega + u, \quad \alpha > 0$$

(a) Design a state feedback $u = -k_1\theta - k_2\omega$ so that the overall closed loop system is stable.

(b) Design an observer to obtain a state estimate $\{\hat{\theta}, \hat{\omega}\}$ from measurements of θ.

(c) Based on the above designs or otherwise, comment on the following,

"... it is hard to imagine that the observer designed for a known input can serve to estimate the state of the process for the purpose of generating the control input."

- Friedland

(a) $A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Charpoly(A) $\equiv$ $s(s+\alpha) - \Omega^2 = 0$

  $\Rightarrow s^2 + \alpha s - \Omega^2 = 0$

  If λ_1, λ_2 are eigenvalues of A,

  $\lambda_1 + \lambda_2 = -\alpha < 0$

  and $\lambda_1 \lambda_2 = -\Omega^2 < 0$

  $\Rightarrow$ one is positive, $\rightarrow$ A is unstable
  other is negative

- Controllable ✓

  $\text{rank}\{[B \quad AB]\} = \text{rank}\left\{\begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix}\right\}$

  $= 2$

- Design

  For $K = [k_1 \quad k_2]$,

  $$A - BK = \begin{bmatrix} 0 & 1 \\ \Omega^2 - k_1 & -\alpha - k_2 \end{bmatrix}$$

  Charpoly$(A-BK) \equiv$

  $s(s + \alpha + k_2) - \Omega^2 + k_1 = 0$

$$\Rightarrow s^2 + (\alpha + k_2)s + (k_1 - \Omega^2) = 0$$

If λ_+, λ_- are eigenvalues of $A - BK$,

$$\lambda_\pm = \frac{-(\alpha + k_2) \pm \sqrt{(\alpha + k_2)^2 - 4(k_1 - \Omega^2)}}{2}$$

$k_2 = 0$, $k_1 > \Omega^2$, say $2\Omega^2$ ensures stability. 2

What region on the $k_1 - k_2$ plane guarantees stability?

(b) $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

- Observable ✓

$$\text{rank}\left\{\begin{bmatrix} C \\ CA \end{bmatrix}\right\} = \text{rank}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= 2$$

- Observer design needs gain L so that $A - LC$ has eigenvalues in left half of the complex plane.

$$A - LC = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} - \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 & 1 \\ \Omega^2 - l_2 & -\alpha \end{bmatrix}$$

Charpoly $(A - LC) \equiv$

$$(s + l_1)(s + \alpha) - (\Omega^2 - l_2) = 0$$

$$s^2 + (l_1 + \alpha)s + (l_1\alpha + l_2 - \Omega^2) = 0$$

$l_1 = \alpha$, $l_2 = \Omega^2$ is one solution

2

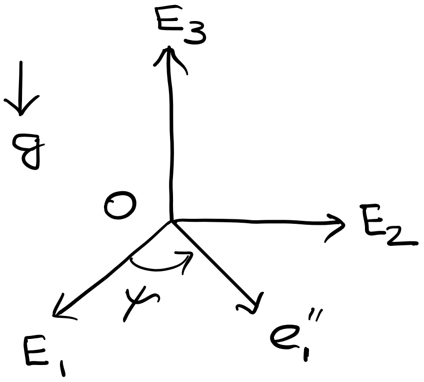
What region on the $l_1 - l_2$ plane guarantees stability?

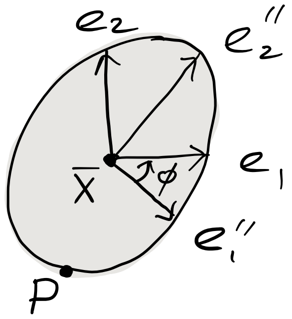
(c) Designs in (a) and (b) were performed separately. (a) did not consider (b) and (b) did not consider (a). It is not clear if they would work together.

1

{3 marks}

4. Consider the following model of a wheel. The wheel is modelled as a rigid circular disk of mass 'm' and radius 'R' that rolls without slipping on a horizontal plane.





$\in (0, \pi)$

θ represents the inclination angle.

$\theta = \frac{\pi}{2} \Rightarrow$ disk is vertical.

Point P is in contact with the horizontal plane.

(x_1, x_2, x_3) are the co-ordinates of the center of mass in $E_1 - E_2 - E_3$.

λ_a and λ_t are some moments of inertia.

The equations of motion are

$$\lambda_t \sin\theta \, \ddot{\psi} = (\lambda_a - 2\lambda_t)\dot{\psi}\,\dot{\theta}\cos\theta + \lambda_a \dot{\theta}\,\dot{\phi}$$

$$\begin{aligned}(\lambda_t + mR^2)\,\ddot{\theta} = (\lambda_t - \lambda_a - mR^2)\,\dot{\psi}^2 \sin\theta\cos\theta \\ - (\lambda_a + mR^2)\,\dot{\psi}\,\dot{\phi}\sin\theta \\ - mgR\cos\theta\end{aligned}$$

$$\begin{aligned}(\lambda_a + mR^2)\cos\theta\,\ddot{\psi} + (\lambda_a + mR^2)\,\ddot{\phi} \\ = (\lambda_a + 2mR^2)\,\dot{\psi}\,\dot{\theta}\sin\theta\end{aligned}$$

$$\begin{aligned}\dot{x}_1 = -R\dot{\psi}\cos\psi\cos\theta + R\,\dot{\theta}\sin\psi\sin\theta \\ - R\dot{\phi}\cos\psi\end{aligned}$$

$$\begin{aligned}\dot{x}_2 = -R\dot{\psi}\sin\psi\cos\theta - R\,\dot{\theta}\cos\psi\sin\theta \\ - R\dot{\phi}\sin\psi\end{aligned}$$

$$\dot{x}_3 = R\,\dot{\theta}\cos\theta$$

Transpose

Define the state vector

$$z = [\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}, \overset{\times}{\dot{x}_1}, \overset{\times}{\dot{x}_2}, \overset{\times}{\dot{x}_3}]'$$

and write the equations in the state-space form $M(z)\,\dot{z} = f(z)$.

$M(z) = ?$, $f(z) = ?$

Look at this, also

https://rotations.berkeley.edu/the-rolling-disk/

$$\begin{array}{ccccccccccc} z = [& \theta, & \phi, & \psi, & \dot{\theta}, & \dot{\phi}, & \dot{\psi}, & x_1, & x_2, & x_3 &]' \\ & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 & Z_8 & Z_9 & \end{array}$$

$$\dot{Z}_1 = Z_4$$

$$\dot{Z}_2 = Z_5$$

$$\dot{Z}_3 = Z_6$$

$$\dot{Z}_7 = -R Z_6 \cos Z_3 \cos Z_1 + R Z_4 \sin Z_3 \sin Z_1 - R Z_2 \cos Z_3$$

$$\dot{Z}_8 = -R Z_6 \sin Z_3 \cos Z_1 - R Z_4 \cos Z_3 \sin Z_1 - R Z_5 \sin Z_3$$

$$\dot{Z}_9 = R Z_4 \cos Z_1$$

$$\lambda_t \sin Z_1 \, \dot{Z}_6 = (\lambda_a - 2\lambda_t) Z_6 Z_4 \cos Z_1 + \lambda_a Z_4 Z_5$$

$$(\lambda_t + mR^2)\, \dot{Z}_4 = (\lambda_t - \lambda_a - mR^2) Z_6^2 \sin Z_1 \cos Z_1 - (\lambda_a + mR^2) Z_5 Z_6 \sin Z_1 - mgR \cos Z_1$$

$$(\lambda_a + mR^2) \cos\theta \, \dot{Z}_6 + (\lambda_a + mR^2)\, \dot{Z}_5 = (\lambda_a + 2mR^2) Z_4 Z_6 \sin Z_1$$

$$M(z) \cdot \dot{z} = f(z)$$

$$f(z) = \begin{bmatrix} z_4 \\ \hline z_5 \\ \hline z_6 \\ \hline (\lambda_t - \lambda_a - mR^2) z_6^2 \sin z_1 \cos z_1 \\ - (\lambda_a + mR^2) z_5 z_6 \sin z_1 \\ - mgR \cos z_1 \\ \hline (\lambda_a + 2mR^2) z_4 z_6 \sin z_1 \\ \hline (\lambda_a - 2\lambda_t) z_6 z_4 \cos z_1 + \lambda_a z_4 z_5 \\ \hline -R z_6 \cos z_3 \cos z_1 + R z_4 \sin z_3 \sin z_1 \\ - R z_2 \cos z_3 \\ \hline -R z_6 \sin z_3 \cos z_1 - R z_4 \cos z_3 \sin z_1 \\ - R z_5 \sin z_3 \\ \hline R z_4 \cos z_1 \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \\ \dot{z}_7 \\ \dot{z}_8 \\ \dot{z}_9 \end{bmatrix}$$

$$M(z) = \left[\begin{array}{c|ccc|c} I_{3\times3} & & 0 & & 0 \\ \hline & \lambda_t + mR^2 & 0 & 0 & \\ 0 & 0 & \lambda_a + mR^2 & (\lambda_a + mR^2)\cos z_1 & 0 \\ & 0 & 0 & \lambda_t \sin z_1 & \\ \hline 0 & & 0 & & I_{3\times3} \end{array}\right]$$

Linearize about equilibrium point. $\rightarrow \dot{z} = 0$

$$M(z)\dot{z} = f(z)$$

$$\Rightarrow f(z) = 0$$

Simplify to analysis of $[z_1, \dots z_6]'$ subsystem as z_7, z_8, z_9 do not "actively participate" in these dynamics.

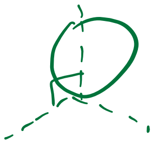
$$\begin{bmatrix} z_4 = 0 \\ z_5 = 0 \\ z_6 = 0 \\ \boxed{(\lambda_t - \lambda_a - mR^2) z_6^2 \sin z_1 \cos z_1} \to 0 \\ -\boxed{(\lambda_a + mR^2) z_5 z_6 \sin z_1} \to 0 \\ - mgR \cos z_1 \\ (\lambda_a + 2mR^2) z_4 z_6 \sin z_1 = 0 \\ (\lambda_a - 2\lambda_t) z_6 z_4 \cos z_1 + \lambda_a z_4 z_5 = 0 \end{bmatrix} = 0 \ ?$$

if $-mgR \cos z_1 = 0$

$z_1 = \pi/2$

(~~odd~~ multiples of $\pi/2$)

One equilibrium point, which is the vertical position.



Equilibrium point,

$$z_0 = \left[\frac{\pi}{2}, \phi_0, \psi_0, 0, 0, 0 \right]'$$

For linearization,

$$z = z_0 + \delta z$$

$$M(z) = M(z_0) + \left.\frac{\partial M}{\partial z}\right|_{z_0} \delta z$$

$$f(z) = \cancel{f(z_0)}^{=0} + \left.\frac{\partial f}{\partial z}\right|_{z_0} \cdot \delta z$$

$$\dot{z} = \cancel{\dot{z}_0}^{=0} + \delta \dot{z}$$

$$\Rightarrow M(z)\, \dot{z} = f(z)$$

becomes

$$\sim \left(M(z_0) + (\,)\delta z \right) \delta \dot{z} = \left.\frac{\partial f}{\partial z}\right|_{z_0} \cdot \delta z$$

$$\sim M(z_0)\, \delta \dot{z} = \left.\frac{\partial f}{\partial z}\right|_{z_0} \delta z$$

↓ 6×6

lower right 3x3

$$\begin{bmatrix} \lambda_t + mR^2 & 0 & 0 \\ 0 & \lambda_a + mR^2 & (\lambda_a + mR^2)\cos z_1 \\ 0 & 0 & \lambda_t \sin z_1 \end{bmatrix}_{z_1 = \pi/2} = \begin{bmatrix} \lambda_t + mR^2 & 0 & 0 \\ 0 & \lambda_a + mR^2 & 0 \\ 0 & 0 & \lambda_t \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial z}\right|_{z_0} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline mgR & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Guess of eigenvalues:

$\lambda_1 = \sqrt{mgR}, \lambda_2 = -\sqrt{mgR}, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, \lambda_6 = 0$

$$\lambda^4(\lambda^2 - mgR) = 0 \ ?$$

For a matrix such as this

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \\ \dot x_3 \\ \dot x_4 \\ \dot x_5 \\ \dot x_6 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \alpha(\neq 0) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$\theta \sim \begin{bmatrix} \dot x_1 \\ \dot x_4 \\ \dot x_2 \end{bmatrix} = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & & 0 \\ \alpha & 0 & & & \\ \hline & & 0 & 1 & \end{array}\right] \begin{bmatrix} x_1 \\ x_4 \\ x_2 \end{bmatrix}$$

$$\begin{array}{c} \phi\sim \\ \\ \psi_1\sim \end{array}\begin{bmatrix} \dot{x}_5 \\ \dot{x}_3 \\ \dot{x}_6 \end{bmatrix} \begin{bmatrix} 0 & \vert & 0 \quad 0 & \vert & 0 \\ \hline & & & \vert\; 0 \;\vert & \\ 0 & & 0 & \vert & 0 \quad 0 \end{bmatrix} \begin{bmatrix} x_5 \\ x_3 \\ x_6 \end{bmatrix}$$

$$\dot{x}_3 = x_6$$
$$\dot{x}_6 = 0$$

Does this remind us of something?

Newton's I Law

$$x_6 = x_6(0)$$

$$x_3 = x_6(0).\underset{\substack{\uparrow \\ t\,e^{0.t}}}{t} + \underset{\substack{\uparrow \\ e^{0.t}}}{x_3(0)}$$