Optimal Control (ELL703) > EndSem Exam

Figure 1

Design a slide of least time.

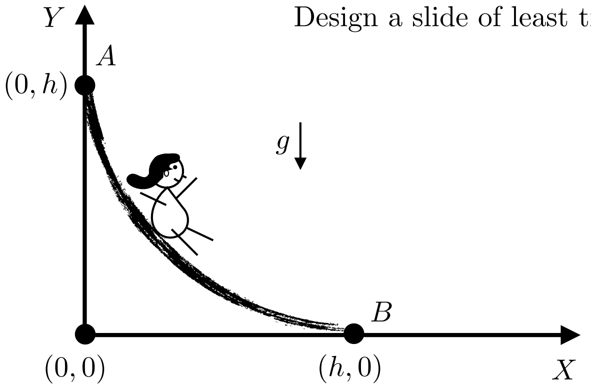


Figure 2

Differential Calculus
> Minimize

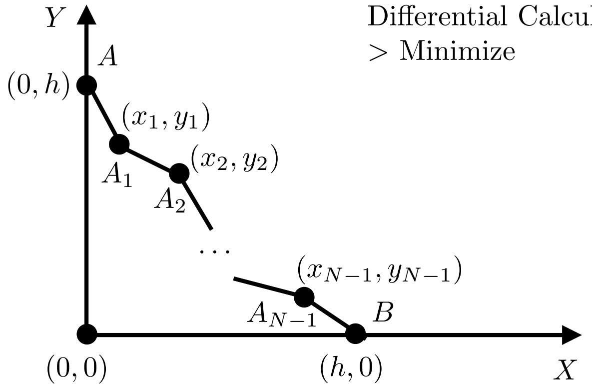


Figure 3

Dynamic Programming > Routing

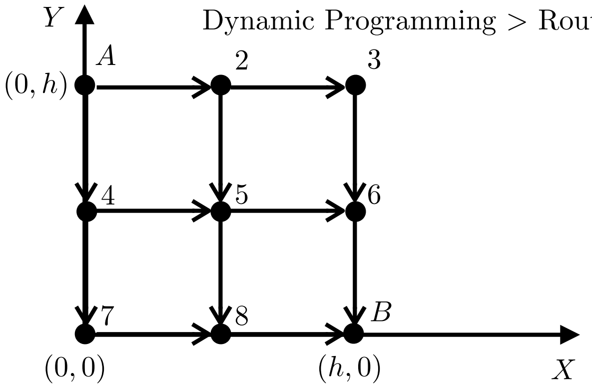


Figure 4

> Hamilton-Jacobi-Bellman Equation

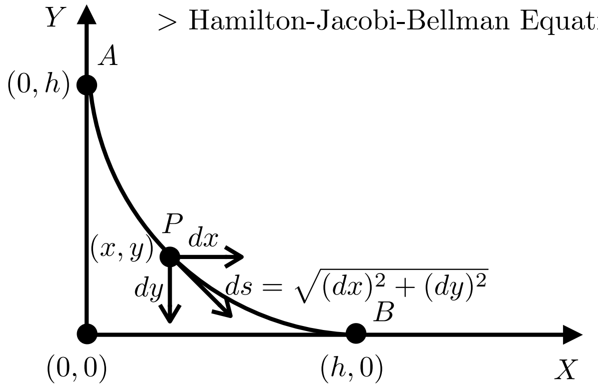


Figure 5

Calculus of Variations
> Euler-Lagrange Equations

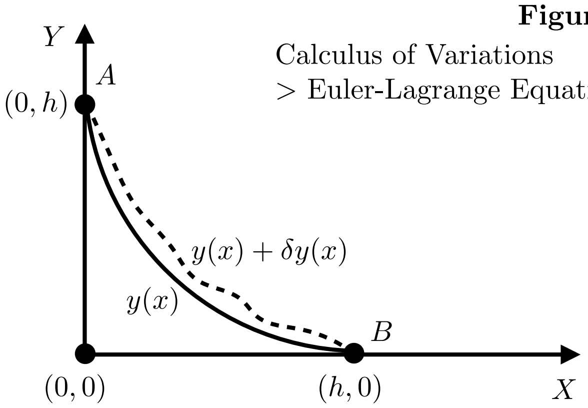


Figure 6

Calculus of Variations
> Lagrange Multipliers
> Hamiltonian

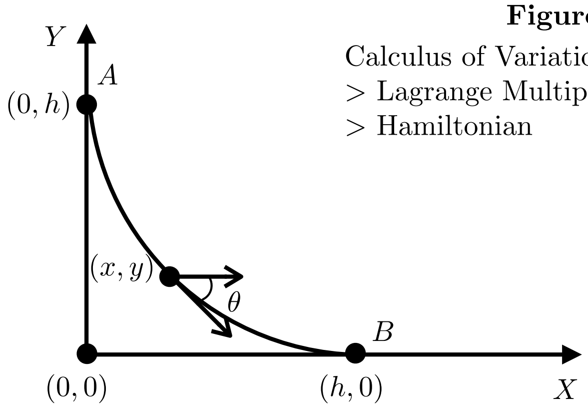


Figure 7

Slide is not a Straight Line.

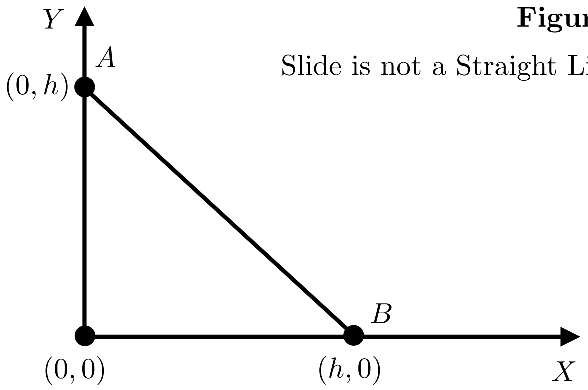
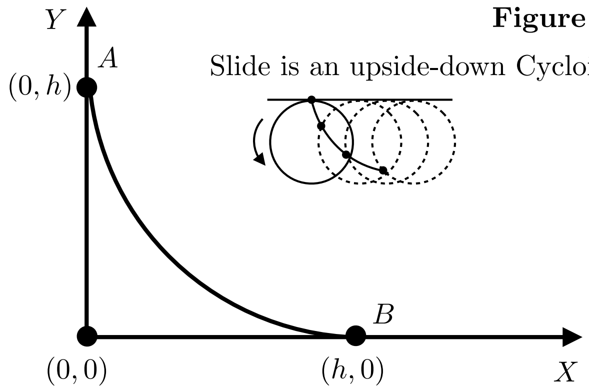


Figure 8

Slide is an upside-down Cycloid.



Optimal Control (ELL703) > EndSem Exam

Instructions

The duration of the exam is 2 hours and the maximum marks are 35.
Each part of the question **Q1** is worth 5 marks. Answer any 7 out of the 8 parts.
Answers with the essence of the 7 C's — Closed, Compact, Complete, Connected, Convergent, Continuous, Convex — will be gratefully appreciated.

Q1. It is desired to **design a slide** for a children's playground (Figure 1). The shape of the slide should be such that the **time** to slide from the top to the bottom is the **least** possible.

Assume that the motion is only under the influence of earth's gravity and that there is no friction. Using the co-ordinates and the notation shown in Figures 1–8, answer the following.

(a) Discretize the desired shape into N line segments AA_1, A_1A_2 ... $A_{N-1}B$ (Figure 2). Denote the co-ordinates of point A_i, $i = 1, 2, \dots N-1$ as (x_i, y_i). Find the length of each segment and the velocity at each point. Assume that the velocity in each segment is the average of the velocities at the endpoints of the segment. Formulate the problem as one of **minimizing, as a function of the variables** (x_i, y_i)**, the total time** it takes to travel from A to B along the N segments.

(b) Discretize space into a 2×2 grid of squares (Figure 3). As in part (a), assume that the velocity in each segment is the average of the velocities at the endpoints of the segment. Find the time it takes to cross each horizontal (left $\to$ right) and each vertical (top $\to$ bottom) segment of the grid. Using **dynamic programming**, find the route of minimum time from A to B.

(c) Show that the **Hamilton-Jacobi-Bellman Equation** that determines the desired shape of the slide $y(x)$ is

$$-\frac{\partial T^*}{\partial x} = \min_{y'} \left\{ \sqrt{\frac{1 + (y')^2}{2g(h-y)}} + \frac{\partial T^*}{\partial y} y' \right\},$$

where T^* is the minimum time from P to B and $y' = dy/dx$ (Figure 4).

(d) Reconsider the formulation in part (a) with each segment of infinitesimal length ($N \to \infty$, Figure 4). Show that the functional to be minimized is $T[y, y'] = \int_0^H \sqrt{\frac{1+(y')^2}{2g(h-y)}} dx$. Using **calculus of variations** (Figure 5), find the necessary conditions (**Euler-Lagrange Equations**) for T to be minimum. How are these conditions related to the conditions in part (c)?

(e) The dynamics of the sliding child may be modelled as $\dot{x} = v \cos\theta$, $\dot{y} = -v \sin\theta$, where $v = \sqrt{2g(h-y)}$ (Figure 6). Minimize the cost functional $\int_0^T dt$ subject to the above system dynamics using the method of **Lagrange Multipliers**. Explicitly state the **Hamiltonian**. Derive the **costate equations**, the **stationarity condition**, and the **boundary conditions**.

(f) Show that **a straight line cannot be the desired shape of the slide** (Figure 7).

(g) Show that the **desired shape of the slide is a cycloid**, with parametric equations are $x(\phi) = c_1(\phi - \sin\phi) + c_2$, $y(\phi) = h - c_1(1 - \cos\phi)$ for constants c_1 and c_2 (Figure 8).

(h) Suppose the **slope of the slide is restricted** to be in the range $[-\tan 60°, 0]$ at each point. Based on the **Pontryagin Principle**, discuss how the shape of the slide could be obtained.

Draft Solutions

It is desired to design a slide for a children's playground. The design specification is for the shape of the slide to be such that the time to slide from the top to the bottom is the least possible.

This minimum-time problem is a version of the famous brachistochrone problem (Greek: brachistos = shortest, chronos = time) posed by Johan Bernoulli. It was solved by Johan Bernoulli himself, his brother Jacob Bernoulli, Leibniz, Isaac Newton, and L'Hospital, possibly through different methods. The problem derives its importance for multiple reasons, including the minimality (!) and elegance of its statement, the non-intuitive nature of the solution (not a straight line!) that can be expressed analytically (a cycloid), as well as the major impetus that these provided to the development of the calculus of variations and related fields. Some of these solutions are discussed in numerous textbooks as well as articles and videos available on the internet. Now, we will explore the solution to this problem.

There are three key assumptions in our exploration of the problem solutions. The first assumption is the presence of gravity on earth (!). This force makes sliding possible. The second assumption is the absence of friction. This is primarily motivated by reasons of simplicity. It is likely that some of the solution approaches will be able to account for this realistic scenario. A third, implicit assumption is that a solution to the problem exists.

(a)

A mathematical representation of the desired shape is a function $y(x)$ that assigns a height y to each point x along the shape. Alternatively, a function $x(y)$ may also be considered. The function $y(x)$ is an infinite-dimensional unknown because of the uncountable infinite number of x's in the interval $[0, h]$. An often useful solution approach is to discretize $y(x)$. This is a form of projection and converts an infinite-dimensional unknown $y(x)$ into a finite dimensional unknown $y_i = y(x_i)$, along suitably chosen x_i and a discrete set of i.

Consider a discretization into N line segments AA_1, A_1A_2 ... $A_{N-1}B$. The points A, A_1, A_2, ... A_{N-1}, and B represent a discrete approximation to the desired shape. The length of each segment is

$$l_i = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}, i = 1, 2, \dots N,$$

where $(x_0, y_0) \equiv (0, h)$ and $(x_N, y_N) \equiv (h, 0)$. The velocity at any point (x_i, y_i) can be obtained from the conservation of energy,

$$mgh = \frac{1}{2}mv_i^2 + mgy_i \Rightarrow v_i = \sqrt{2g(h - y_i)}.$$

We assume that the velocity in each segment is the average velocity at the end-points of the segment. If we did not have an assumption along these lines, the slide would not start! The time in each segment t_i is the ratio of the length and the velocity. Therefore, the function to be minimized is

$$T = \sum_{i=1}^{N} t_i = \sum_{i=1}^{N} \frac{l_i}{(v_i + v_{i-1})/2},$$

which is a function of (x_i, y_i).

This is minimization problem is a problem in differential calculus, albeit of the multivariable variety. The necessary condition is for the partial derivatives of T with respect to the $2N - 2$ variables ($N - 1$ each off x_i and y_i) to be equal to zero. The solution of these $2N - 2$ equations may be obtained numerically, for example the Newton's method or its variants. Alternatively, a dynamic programming perspective could be adopted to solve the problem step-by-step. It is likely that these approaches are not amenable to an exact solution and may suffer from numerical errors. It would be interesting to use an interval Newton Algorithm to solve this problem to rigorously find all solutions in the interval box $[0, h]^{N-1} \times [0, h]^{N-1}$. In all of the above cases, the hope is that as $N \to \infty$, the solution would converge to the desired shape. For $N = 2$, the necessary condition should give the Snell's law of refraction, for a ray of light that crosses from one medium to another at the boundary defined by $y = y_1$. We shall return to this point in part (e).

(b)

Instead of discretizing the curve, we can discretise space itself. Let us consider a very coarse discretisation in the form of a 2×2 grid of squares.

The length of each horizontal and vertical segment is $h/2$. As in part (a), we have to explicitly assume something about the velocity starting from A to get the motion started. We assume, therefore, that the velocity of each segment is the average of the velocities of the endpoints. An implicit assumption is that the motion is directed: left $\to$ right and top $\to$ bottom. Accordingly, we can write the times to travel each segment as,

$$\begin{aligned}
t_{A2} &= \frac{h/2}{0} = \infty = t_{23}, \\
t_{A4} &= \frac{h/2}{(0 + \sqrt{2gh/2})/2} = \frac{h/2}{\sqrt{gh}/2} = t_{25} = t_{36}, \\
t_{45} &= \frac{h/2}{(\sqrt{2gh/2} + \sqrt{2gh/2})/2} = \frac{h/2}{\sqrt{gh}} = t_{56}, \\
t_{47} &= \frac{h/2}{(\sqrt{2gh} + \sqrt{2gh/2})/2} = \frac{h/2}{\sqrt{gh}(1+\sqrt{2})/2} = t_{58} = t_{69}, \\
t_{78} &= \frac{h/2}{(\sqrt{2gh} + \sqrt{2gh})/2} = \frac{h/2}{\sqrt{2gh}} = t_{89}.
\end{aligned}$$

This setup is very similar to the routing problem in dynamic programming. This can be used to find the path of minimum time from A to B,

$$\begin{aligned}
t^*_{AB} &= \min t_{AB} = \min\{t_{A2} + t^*_{2B}, t_{A4} + t^*_{4B}\} = t_{A4} + \min\{t_{45} + t^*_{5B}, t_{47} + t^*_{7B}\} \\
&= t_{A4} + \min\{t_{45} + \min\{t_{58} + t_{8B}, t_{56} + t_{6B}\}, t_{47} + t_{78} + t_{8B}\} \\
&= \frac{h/2}{\sqrt{gh}}\Big(2 + \min\{1 + \min\{\frac{2}{\sqrt{2}+1} + \frac{1}{\sqrt{2}}, 1 + \frac{2}{\sqrt{2}+1}\}, \frac{2}{\sqrt{2}+1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\}\Big).
\end{aligned}$$

Consideration of this expression shows that the minimum time path is $A \to 4 \to 7 \to 8 \to B$. Because $t_{A2} = \infty$, there are effectively only two paths from A to B, which can be evaluated directly as well.

The dependence on h and g seems justifiable. Of course, the 2×2 resolution is too coarse to conclusively obtain the desired shape. Based on this exercise, it is unclear if a straight line is a possible solution. Nonetheless, an increase in resolution via an $M \times M$ grid where $M = 3, 4, \ldots$ has the potential of conceptually simple and practical way to compute the desired shape. This formulation also highlights that at each node a decision on the next direction to take has to be made.

(c)

The decision making process in part (b) can also be formulated in the continuous setting, in the limit of an infinitely fine grid. The decision input, or control, at each node is the slope $y' = dy/dx$.

Let $T(x, y, y')$ denote the time to travel from an intermediate point P with co-ordinates (x, y) to B. Then,

$$\begin{aligned} T(x, y, y') &= T(x + \Delta x, y + \Delta y, y') + \Delta t, \\ \Rightarrow T^*(x, y, y') &= \min_{y'_{P \to B}} \{T(x + \Delta x, y + \Delta y, y') + \Delta t\}, \end{aligned}$$

where the time from P to B is broken into the time from P to a point close to it, say $P + \Delta P$, with co-ordinates $(x + \Delta x, y + \Delta y)$ and the time from $P + \Delta P$ to B. Now

$$\begin{aligned} T^*(x, y, y') &= \min_{y'_{P \to P+\Delta P}} \min_{y'_{P+\Delta P \to B}} \left\{T(x + \Delta x, y + \Delta y, y') + \sqrt{\frac{1 + (y')^2}{2g(h - y)}}\Delta x\right\}, \\ &= \min_{y'_{P \to P+\Delta P}} \left\{T^*(x + \Delta x, y + \Delta y) + \sqrt{\frac{1 + (y')^2}{2g(h - y)}}\Delta x\right\}. \end{aligned}$$

Expand $T^*(x + \Delta x, y + \Delta y, y')$ in a Taylor Series around (x, y) and assume that $\Delta x \to dx$ and $\Delta y \to dy$, which are infinitesimal quantities.

$$\begin{aligned} \Rightarrow T^*(x, y, y') &= \min_{y'(atP)} \left\{T^*(x, y) + \frac{\partial T^*}{\partial x}dx + \frac{\partial T^*}{\partial x}dy + \sqrt{\frac{1 + (y')^2}{2g(h - y)}}dx\right\}, \\ \Rightarrow -\frac{\partial T^*}{\partial x} &= \min_{y'} \left\{\sqrt{\frac{1 + (y')^2}{2g(h - y)}} + \frac{\partial T^*}{\partial x}y'\right\}. \end{aligned}$$

This is the desired Hamilton-Jacobi-Bellman equation.

In principle, this partial differential equation is our solution. There are, however, very few such equations that can be exactly solved. Typically, solutions to such partial differential equations have to be obtained by a discretization approach. It is just that the discretization is performed after such an equation is obtained rather than directly, as in parts (a) and (b). Nonetheless, this approach is important in providing a recipe to choose the optimal decision/ control input.

(d)

Dynamic programming, either in a discrete context or in a continuous setting, aims at arriving at the best decision at each step or location. In contrast to such a microscopic perspective, a telescopic perspective, harking back to part (a), albeit in continuous domain, may be adopted. This, in essence, is the approach of the calculus of variations. If dynamic programming is akin to specifying a curve as an envelope of tangents, calculus of variations seeks to specify the same curve from a member of a family of curves. This latter concept needs the ideas of a functional — a function of a function — and a variation.

The overall time $T[y, y'] = \int_A^B dt$ is a functional that is to be minimized over all paths possible. As in part (c), $dt = \sqrt{\frac{1+(y')^2}{2g(h-y)}}dx$. Denote $L(y, y') = \sqrt{\frac{1+(y')^2}{2g(h-y)}}$. Therefore, $T[y, y'] = \int_0^h L(y, y')dx$ is the functional to be minimized.

A necessary condition for T to be minimum is that its first variation $\delta T = 0$. To compute this, we consider a variation in $y(x) \to y(x) + \epsilon\eta(x)$, where $\eta(x)$ is a well-behaved function with $\eta(0) = 0 = \eta(h)$. Correspondingly, $y'(x) \to y'(x) + \epsilon\eta'(x)$. Consider the time functional over a set of neighbouring paths,

$$T[y + \epsilon\eta, y' + \epsilon\eta'] = \int_0^h L(y + \epsilon\eta, y' + \epsilon\eta')dx,$$

and expand in a Taylor Series around $\epsilon = 0$,

$$\Rightarrow T[y + \epsilon\eta, y' + \epsilon\eta'] = T[y, y'] + \int_0^h \left[\frac{\partial L}{\partial y}\epsilon\eta + \frac{\partial L}{\partial y'}\epsilon\eta'\right] dx + O(\epsilon^2).$$

The first variation is the $O(\epsilon)$ term in the difference $T[y + \epsilon\eta, y' + \epsilon\eta'] - T[y, y']$,

$$\delta T = \int_0^h \left[\frac{\partial L}{\partial y}\epsilon\eta + \frac{\partial L}{\partial y'}\epsilon\eta'\right] dx.$$

Integrate the second term by parts,

$$\Rightarrow \delta T = \frac{\partial L}{\partial y'}\epsilon\eta(x)|_0^h + \int_0^h \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right]\epsilon\eta dx.$$

As $\eta(o) = 0 = \eta(h)$, the first variation $\delta T = 0$ for all variations $\epsilon\eta(x)(= \delta y)$ if,

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0.$$

These are the Euler-Lagrange equations. For the above L, these equations may be simplified to $1 + (y')^2 - 2y''(H - y) = 0$. The boundary conditions are $y(0) = h$ and $y(h) = 0$. As expected, there are two boundary conditions to completely specify the solution of this second order differential equation. This is the necessary condition.

Interestingly, this condition is the same as in part (c). To see this, let us revisit the condition derived on the right-hand side in part (c),

$$\min_{y'}\left\{\sqrt{\frac{1 + (y')^2}{2g(h - y)}} + \frac{\partial T^*}{\partial x}y'\right\} \Rightarrow \frac{\partial L}{\partial y'} + \frac{\partial T^*}{\partial y} = 0.$$

At this equation,

$$-\frac{\partial T^*}{\partial x} = L + \frac{\partial T^*}{\partial y}y'.$$

Now take the derivative of the former equation with respect to x and a partial derivative of the latter equation with respect to y:

$$\frac{\partial L}{\partial y'} + \frac{\partial T^*}{\partial y} = 0 \;\Rightarrow\; \frac{d}{dx}\frac{\partial L}{\partial y'} + \frac{\partial^2 T^*}{\partial x \partial y} + \frac{\partial^2 T^*}{\partial y^2}y' = 0.$$

$$L + \frac{\partial T^*}{\partial y}y' + \frac{\partial T^*}{\partial x} = 0 \;\Rightarrow\; \frac{\partial L}{\partial y} + \frac{\partial^2 T^*}{\partial x \partial y} + \frac{\partial^2 T^*}{\partial y^2}y' = 0.$$

A comparison of these two equations results in the Euler-Lagrange equation,

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0.$$

This is interesting because the results of two different frameworks coincide!

(e)

Instead of prioritising the minimisation problem, with the constraints already embedded in the functional to be minimized, it is possible to emphasize the constraints, in the form of system dynamics, and seek to minimize a property of the system dynamics. This change in emphasis is one of the differences between calculus of variations and optimal control. The constraints and the functional to be minimized are combined using the method of Lagrange multipliers.

For the system dynamics $\dot{x} = v\cos\theta$, $\dot{y} = -v\sin\theta$, where $v = \sqrt{2g(h-y)}$, and the cost functional $\int_0^T dt = T$, consider the augmented cost,

$$J = \int_0^T [1 + \lambda_1(v\cos\theta - \dot{x}) + \lambda_2(-v\sin\theta - \dot{y})]dt,$$

where λ_1 and λ_2 are the two Lagrange multipliers. Minimization of this augmented cost functional is as ordained by the calculus of variations. The variations are $x(t) \to x(t) + \epsilon\eta_1(t)$, $y(t) \to y(t) + \epsilon\eta_2(t)$, $\theta(t) \to \theta(t) + \epsilon\eta_3(t)$, $T \to T + \epsilon\eta_4$.

As in part (d), we seek to set the first variation $\delta J = 0$. For this we calculate,

$$\begin{aligned}
J[x+\epsilon\eta_1, y+\epsilon\eta_2, \theta+\epsilon\eta_3] &= \int_0^{T+\epsilon\eta_4} [1 + \lambda_1(\sqrt{2g(h-y-\epsilon\eta_2)}\cos(\theta+\epsilon\eta_3) - \dot{x} - \epsilon\dot{\eta_1}) \\
&\quad +\lambda_2(-\sqrt{2g(h-y-\epsilon\eta_2)}\sin(\theta+\epsilon\eta_3) - \dot{y} - \epsilon\dot{\eta_2})]dt, \\
&= [1 + \lambda_1 v\cos\theta - \lambda_2 v\sin\theta]|_T\epsilon\eta_4 - \lambda_1\dot{x}|_T\epsilon\eta_4 - \lambda_2\dot{y}|_T\epsilon\eta_4 \\
&\quad + \int_0^T [1 + \lambda_1(\sqrt{2g(h-y-\epsilon\eta_2)}\cos(\theta+\epsilon\eta_3) - \dot{x} - \epsilon\dot{\eta_1}) \\
&\quad +\lambda_2(-\sqrt{2g(h-y-\epsilon\eta_2)}\sin(\theta+\epsilon\eta_3) - \dot{y} - \epsilon\dot{\eta_2})]dt, \\
&= [1 + \lambda_1 v\cos\theta - \lambda_2 v\sin\theta]|_T\epsilon\eta_4 - \lambda_1\dot{x}|_T\epsilon\eta_4 - \lambda_2\dot{y}|_T\epsilon\eta_4 \\
&\quad +J[x,y,\theta] + \int_0^T [\lambda_1\frac{\partial v}{\partial y}\cos\theta\epsilon\eta_2 + \lambda_1 v(-\sin\theta)\epsilon\eta_3 \\
&\quad -\lambda_2\frac{\partial v}{\partial y}\sin\theta\epsilon\eta_2 - \lambda_1\epsilon\dot{\eta_1} - \lambda_2 v(\cos\theta)\epsilon\eta_3 - \lambda_2\epsilon\dot{\eta_2}]dt,
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta J &= [1 + \lambda_1 v\cos\theta - \lambda_2 v\sin\theta]|_T\epsilon\eta_4 \\
&\quad -\lambda_1(\dot{x}|_T\epsilon\eta_4 + \epsilon\eta_1(T)) + \lambda_1\epsilon\eta_1(0) - \lambda_2(\dot{y}|_T\epsilon\eta_4 + \epsilon\eta_2(T)) + \lambda_2\epsilon\eta_2(0) \\
&\quad + \int_0^T [\dot{\lambda}_1\epsilon\eta_1 + (\lambda_1\frac{\partial v}{\partial y}\cos\theta - \lambda_2\frac{\partial v}{\partial y}\sin\theta + \dot{\lambda}_2)\epsilon\eta_2 + (-\lambda_1 v\sin\theta - \lambda_2 v\cos\theta)\epsilon\eta_3]dt,
\end{aligned}$$

Setting the co-efficients of each differential/ variation to zero gives the desired costate equations,

$$\begin{aligned}
\dot{\lambda}_1 &= 0, \\
-\dot{\lambda}_2 &= \lambda_1\frac{\partial v}{\partial y}\cos\theta - \lambda_2\frac{\partial v}{\partial y}\sin\theta,
\end{aligned}$$

the stationarity condition $-\lambda_1 v\sin\theta - \lambda_2 v\cos\theta = 0$, and the boundary condition $[1+\lambda_1 v\cos\theta - \lambda_2 v\sin\theta]|_T = 0 = H|_T$, where $H = 1 + \lambda_1 v\cos\theta - \lambda_2 v\sin\theta$ is the Hamiltonian.

This definitely has more mathematical expressions than in part (d)! One interesting aspect that emerges from simplifying these equations further is a version of the Snell's Law. This is because the costate λ_1 is constant. As H is not an explicit function of time, $\partial H/\partial t = 0$ and the boundary condition implies that $H = 0$ at all times along the optimal trajectory. This equation combined with the stationarity condition can be used to solve for λ_1, which is a constant, in terms of the velocity v and θ. The relation is $\cos\theta/v =$ constant. An inspection of the way θ is defined confirms that this is indeed Snell's Law. In fact, Johan Bernoulli's original solution treated the sliding particle as light, which was known to take the path of least time (Fermat's Principle). The desired shape was computed as the path of light that passed through a series of different media with infinitesimal thickness in each of which the velocity was $v = \sqrt{2g(h-y)}$. As we know that light bends when crossing from one medium into another, where it has different velocity, this is a hint that the desired shape is not a straight line.

(f)

A common guess for the desired shape is that it is a straight line. A possible source of this guess is that the straight line is the shortest distance (in a Euclidean space, another calculus of variations problem!). However this is not the case, as can be shown through some of the above approaches.

To see this from part (d), consider the equation of a straight line joining A and B, $x+y=h$. This does not satisfy the necessary condition, $1+(y')^2-2y''(h-y)=0$, because $1+(-1)^2 \neq 0$. Therefore, a straight line is not the desired shape.

The same conclusion can be obtained from some of the other approaches as well. The solution in part (e) necessitates that the desired shape satisfies the relation $\cos\theta/v = \text{constant}$, where v depends on y. As the slope of the desired shape depends on θ, which changes depending on y, the desired shape cannot be a straight line, which has a constant slope. The solution in part (c) has been shown to be the same as in part (d), so the above argument applies.

The solution in part (b) also hinted on a non-straight line solution, but as the solution approach was a discrete approximation, a rigorous conclusion along these lines cannot be made. Despite this, solution in part (b) provides good intuition into why a straight line might not be the desired shape: A larger vertical displacement in the initial part of the curve provides speed that can minimize the time. Of course, this larger speed needs to be balanced by the possibly larger distance that needs to be travelled. A tradeoff between the speed and the distance gives the desired shape.

(g)

If the straight line is not the desired shape, then what is? Is it an arc of a circle (Galileo's guess) or something else? The solution approaches in parts (c), (d), and (e) may be explicitly solved to give the desired shape. The shape is a cycloid, a curve obtained by tracing the path of a a point on the rim of a rolling circle. Here, we content ourselves with verifying that the equations of the cycloid satisfy the derived conditions.

The parametric equations of an upside down cycloid are $x(\phi) = c_1(\phi - \sin\phi) + c_2$, $y(\phi) = h - c_1(1-\cos\phi)$ for constants c_1 and c_2. Let us check if these satisfy the equation of the curve given by the necessary condition $1+(y')^2-2y''(h-y)=0$.

$$\begin{aligned} y' &= \frac{dy}{dx} = \frac{dy/d\phi}{dx/d\phi} = -\frac{\sin\phi}{1-\cos\phi}, \\ y'' &= \frac{dy'}{dx} = \frac{dy'/d\phi}{dx/d\phi} = \frac{1}{c_1(1-\cos\phi)^2}. \end{aligned}$$

Using these, $1+(y')^2-2y''(h-y)=0$. Therefore, the cycloid above can be the desired shape.

To see how the cycloid corresponds to a circle, technically called its evolute, consider the parametric equations above. The components $(-c_1\sin\phi, c_1\cos\phi)$ represent the motion along a circle. The components $(c_1\phi, h-c_1)$ represent an offset of the center of the circle from the origin. The y-offset is fixed, but the x-offset continually increases with ϕ.

(h)

If the slope of the slide is restricted, as could happen to limit the motion that can be undergone for safety reasons, Pontryagin's principle could be used. Similar to the Hamilton-Jacobi-Bellman equation, the optimal slope $y' = \tan\theta$ at each point would be the argument that minimizes the Hamiltonian $H = L(y, y') + \lambda_1 v\cos\theta - \lambda_2 v\sin\theta$. The range would give the allowable values of the slope over which the minimization is to be performed.