ELL 703 > MidSem Exam Solutions

1. $$J^* = \min_{u_0, u_1} J$$
$$= \min_{u_0, u_1} [(x_2+2)^2 + u_1^2 + u_0^2]$$
$$= \min_{u_0} \min_{u_1} [(x_2+2)^2 + u_1^2 + u_0^2]$$
$$= \min_{u_0} [\underbrace{\min_{u_1} [(x_2+2)^2 + u_1^2]}_{J_1} + u_0^2]$$

Minimize $J_1 = (\frac{1}{2}x_1 \overset{+2}{\,} + u_1)^2 + u_1^2$

For minima, $\frac{\partial J_1}{\partial u_1} = 0$ & $\frac{\partial^2 J_1}{\partial u_1^2} > 0$.

$$\frac{\partial J_1}{\partial u_1} = 2(\frac{1}{2}x_1 + u_1 + 2) \cdot 1 + 2u_1.$$
$$\Rightarrow x_1 + 4u_1 + 4 = 0,$$
$$\Rightarrow u_1^* = -1 - \frac{x_1}{4}.$$
$$\frac{\partial^2 J_1}{\partial u_1} = 4 > 0 \Rightarrow J_1^* = 2\left(1 + \frac{x_1}{4}\right)^2.$$

$$J = J_1^* + u_0^2$$
$$= 2\left(1 + \frac{1}{8}x_0 + \frac{1}{4}u_0\right)^2 + u_0^2$$

For minima, $\frac{\partial J}{\partial u_0} = 0$ & $\frac{\partial^2 J}{\partial u_0^2} > 0$.

$$\frac{\partial J}{\partial u_0} = 4\left(1 + \frac{1}{8}x_0 + \frac{1}{4}u_0\right)\frac{1}{4} + 2u_0.$$
$$\Rightarrow \cancel{5 + \frac{1}{2}x_0}\ 1 + \frac{1}{8}x_0 + \frac{9}{4}u_0 = 0,$$
$$\Rightarrow u_0^* = -\frac{4}{9} - \frac{x_0}{18}.$$
$$\frac{\partial^2 J}{\partial u_0} = \frac{1}{4} + 2 > 0. \Rightarrow J_0^* = 2\left(1 + \frac{1}{8}x_0 - \frac{1}{9} - \frac{x_0}{72}\right)^2 + \left(\frac{4}{9} + \frac{x_0}{18}\right)^2$$
$$= 2\left(\frac{8}{9} + \frac{8}{72}x_0\right)^2 + \left(\frac{4}{9} + \frac{x_0}{18}\right)^2$$

Given $x_0 = 10$,

$$u_0^* = -\frac{4}{9} - \frac{10}{18} = -1, \quad u_1^* = -1 - \frac{4}{4} = -2$$
$$x_1 = \frac{1}{2} \times 10 - 1 = 4, \quad x_2 = \frac{1}{2} \times 4 - 2 = 0$$

and $J^* = 4 + 4 + 1 = 9.$

2. 16 because

$t^*_{19} = \min t_{19}$

$= \min\{t_{12}+t^*_{29},\ t_{13}+t^*_{39}\}$

$= \min\{4+\min t_{29},\ 5+\min t_{39}\}$

$= \min\{4+\min\{t_{24}+t^*_{49},\ t_{25}+t^*_{59}\},$
$\quad 5+\min\{t_{35}+t^*_{59},\ t_{36}+t^*_{69}\}\}$

$= \min\{4+\min\{3+\min t_{49},\ 5+\min t_{59}\},$
$\quad 5+\min\{3+\min t_{59},\ 5+\min t_{69}\}\}$

$= \min\{4+\min\{3+t_{47}+t_{79},\ 5+\min\{t_{57}+t_{79},\ t_{58}+t_{89}\}\},$
$\quad 5+\min\{3+\quad,\ 5+t_{68}+t_{89}\}$

$= \min\{4+\min\{3+7+2,\ 5+\min\{8+2,\ 6+3\}\},$
$\quad 5+\min\{3+\quad,\ 5+9+3\}\}$

$= \min\{4+\min\{12,\ 5+\min\{10,9\}\},$
$\quad 5+\min\{3+\quad,\ 17\}\}$

$= \min\{4+\min\{12,14\},$
$\quad 5+\min\{3+9,\ 17\}\}$

$= \min\{4+12,\ 5+12\}$

$= \min\{16, 17\}$

$= 16.$

Path: $1 \to 2 \to 4 \to 7 \to 9$.

3.

$$J^*(t, x_t) = \min_{u} \left\{ \Phi(T, x_T) + \int_{t}^{T} L(\tau, x, u)\, d\tau \right\}$$

$$= \min_{u} \left\{ \Phi(T, x_T) + \int_{t}^{t+\Delta t} L(\tau, x, u)\, d\tau + \int_{t+\Delta t}^{T} L(\tau, x, u)\, d\tau \right\}$$

$$= \min_{\substack{u(\tau) \\ t \le \tau \le T}} \left\{ \Phi(T, x_T) + \int_{t+\Delta t}^{T} L(\tau, x, u)\, d\tau + \int_{t}^{t+\Delta t} L(\tau, x, u)\, d\tau \right\}$$

$$= \min_{\substack{u(\tau) \\ t \le \tau \le t+\Delta t}} \cdot \min_{\substack{u(\tau) \\ t+\Delta t \le \tau < T}} \left\{ \Phi(T, x_T) + \int_{t+\Delta t}^{T} L(\tau, x, u)\, d\tau + \int_{t}^{t+\Delta t} L(\tau, x, u)\, d\tau \right\}$$

$$= \min_{\substack{u(\tau) \\ t \le \tau < t+\Delta t}} \left\{ J^*(t+\Delta t, x_{t+\Delta t}) + \int_{t}^{t+\Delta t} L(\tau, x, u)\, d\tau \right\}$$

$$= \min_{\substack{u(\tau) \\ t \le \tau < t+\Delta t}} \left\{ J^*(t+\Delta t, x_t + \Delta t f) + \int_{t}^{t+\Delta t} L(\tau, x, u)\, d\tau \right\}$$

$\downarrow$ Taylor Series

$$= \min_{\substack{u(\tau) \\ t \le \tau < t+\Delta t}} \left\{ J^*(t, x_t) + \frac{\partial J^*}{\partial t} \Delta t + \left(\frac{\partial J^*}{\partial x}\right)^T f(t, x, u) \Delta t + O(\tau) + L(t, x_t, u_t) \Delta t \right\}$$

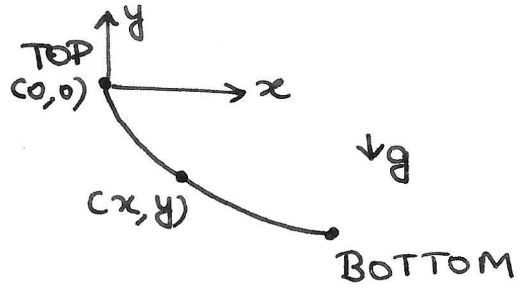
As $\Delta t \to 0$ (after cancelling $J^*(t, x_t)$ and dividing by Δt)

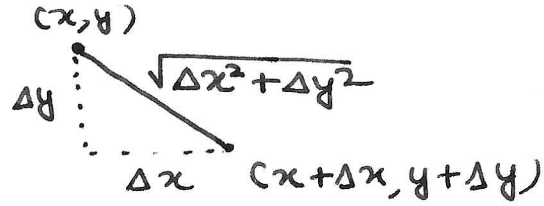
doesn't depend on u_t

$$-\frac{\partial J^*}{\partial t} = \min_{u_t} \left\{ L(t, x_t, u_t) + \left(\frac{\partial J^*}{\partial x}\right)^T f(t, x, u) \right\}$$

as required.

4.





Assumption : Initial velocity is zero.

$$\Rightarrow \quad \frac{1}{2} m v^2 = -mgy \qquad (y<0)$$

where v is the velocity at ~~the~~ point (x,y)

$= \cancel{\frac{dy}{dt}}$

$\therefore$ Time to go from (x,y) to $(x+\Delta x, y+\Delta y)$,

$$\Delta t\,(x,y) = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\sqrt{-2gy}} \rightarrow$$

Minimum time from (x,y) to bottom

$= \Delta t\,(x,y)$ + Minimum time from $(x+\Delta x, y+\Delta y)$ to bottom

This is the problem formulation in terms of dynamic programming.

One way to solve is to solve the corresponding HJB PDE formed from the minimum time cost and system dynamics.

Another way is to discretize the space into a square grid and convert the problem into a routing problem.

5. $\overset{\min}{\cancel{\max}} \sum_{i=1}^{n} x_i^2$

sub $\sum_{i=1}^{n} x_i = c$

Let $J = \sum_{i=1}^{n} x_i^2$

$$J^* = \min_{\sum_{i=1}^{n} x_i = c} \sum_{i=1}^{n} x_i^2 ,$$

$$= \min_{x_1} \min_{\sum_{i=2}^{n} x_i = c - x_1} \left\{ x_1^2 + \sum_{i=2}^{n} x_i^2 \right\},$$

$$= \min_{x_1} \left\{ x_1^2 + \min_{\sum_{i=2}^{n} x_i = c - x_1} \sum_{i=2}^{n} x_i^2 \right\}.$$

This suggests the subproblems,

$$\min \quad J_k = \sum_{i=k}^{n} x_i^2$$

$$\text{sub} \quad \sum_{i=k}^{n} x_i = c - \sum_{i=1}^{k-1} x_i$$

<u>$k = n$</u> $\quad J_n = x_n^2 = \left(c - \sum_{i=1}^{n-1} x_i\right)^2 = J_n^*$

The choice of x_n is fixed.

<u>$k = n-1$</u> $\quad J_{n-1} = x_{n-1}^2 + \left(c - \sum_{i=1}^{n-1} x_i\right)^2$

$$J_{n-1}^* = \min_{x_{n-1}} \left\{ x_{n-1}^2 + \left(c - \sum_{i=1}^{n-1} x_i\right)^2 \right\}$$

For minima,

$$\frac{\partial J_{n-1}}{\partial x_{n-1}} = 2x_{n-1} + 2\left(c - \sum_{i=1}^{n-1} x_i\right)(-1) = 0$$

$$\Rightarrow \quad 2x_{n-1}^* = c - \sum_{i=1}^{n-2} x_i$$

$$\Rightarrow \quad x_{n-1}^* = \frac{1}{2}\left(c - \sum_{i=1}^{n-2} x_i\right)$$

$$\frac{\partial^2 J_{n-1}}{\partial x_{n-1}^2} = 4 > 0$$

$$J_{n-1}^* = \frac{1}{2}\left(c - \sum_{i=1}^{n-2} x_i\right)^2$$

<u>$k = n-2$</u> $\quad J_{n-2} = x_{n-2}^2 + J_{n-1}$

$$J_{n-2}^* = \min_{x_{n-2}} \left\{ x_{n-2}^2 + J_{n-1}^* \right\}$$

$$\frac{\partial J_{n-2}}{\partial x_{n-2}} = 2x_{n-2} + \left(c - \sum_{i=1}^{n-2} x_i\right)(-1) = 0$$

$$\Rightarrow x^*_{n-2} = \frac{1}{3}\left(c - \sum_{i=1}^{n-3} x_i\right).$$

$$\frac{\partial^2 J_{n-2}}{\partial x_{n-2}^2} = 3 > 0$$

$$J^*_{n-2} = \frac{1}{3}\left(c - \sum_{i=1}^{n-3} x_i\right)^2.$$

The general pattern, which could be proved using induction is

$$J^*_k = \frac{1}{n-k+1}\left(c - \sum_{i=1}^{k-1} x_i\right)^2$$

$$x^*_k = \frac{1}{n-k+1}\left(c - \sum_{i=1}^{k-1} x_i\right)$$

$$\therefore \quad x^*_1 = \frac{c}{n}$$

$$x^*_2 = \frac{1}{n-1}\left(c - \frac{c}{n}\right) = \frac{c}{n}$$

and, again using induction,

$$x^*_k = \frac{c}{n}, \quad k = 1, 2, \ldots n.$$