1. [Żak] **Variation of a Functional.** If $v = \int_0^1 (2x^2(t) + x(t))dt$, find δv.

2. [Żak] **Extrema of a Functional.** If $v = \int_0^{\pi/2} ((\dot{x})^2 - x^2)dt$ with $x(0) = 0$ and $x(\pi/2) = 1$, find $x(t)$ where $\delta v = 0$.

3. [Lewis] Show that the shortest distance between two points is a straight line.

4. [Feynman/ Levi/ Lewis] **The Principle of Least Action.** Read Chapter 19 in Volume 2 of Feynman's Lecture in Physics available at,
https://www.feynmanlectures.caltech.edu/II_19.html
a. Define the Lagrangian as the difference of a particles kinetic and potential energies $L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - U(x)$ and the action as $J[x] = \int_0^T L(x, \dot{x})dt$. The initial and final positions of the particle are fixed. Show that $\delta J = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}$. These are called the Euler-Lagrange equations.
b. Substitute the definition of the Lagrangian in the Euler-Lagrange equations to recover Newton's Law $m\ddot{x} = -\frac{\partial U}{\partial x}$.
c. For any solution x of the Euler-Lagrange equation show that $\dot{x}\frac{\partial L}{\partial \dot{x}} - L$ is a constant. This is Noether's theorem and a statement of Conservation of Energy.
d. Find the extrema of the above action subject to the constraint $\dot{x} = u$ using a Lagrange multiplier λ. This would involve defining a Hamiltonian $H = L + \lambda u$ in an intermediate step. Show that the stae and costae equations obtained by setting $\delta J = 0$ are the Hamilton's equations of motion $\dot{x} = \frac{\partial H}{\partial \lambda}$ and $-\dot{\lambda} = \frac{\partial H}{\partial x}$. Therefore, the state and costate equations may be viewed as a generalization of Hamilton's equations of motion.
e. Verify the above for a harmonic oscillator $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$.