FINITE ELEMENT METHOD (FEM): AN OVERVIEW

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ANALYTICAL / MATHEMATICAL SOLUTIONS

- **RESULTS AT INFINITE LOCATIONS**
- **CONTINUOUS SOLUTIONS**
- **FOR SIMPLIFIED SITUATIONS ONLY**
- **EXACT SOLUTION**

NUMERICAL (FEM) SOLUTIONS

- **APPROXIMATE SOLUTIONS**
- **VALUES AT DISCRETE LOCATIONS**
- **FOR COMPLEX**
  GEOMETRY
  MATERIAL PROPERTIES
  LOADING
  BOUNDARY CONDITIONS

THE FINITE ELEMENT METHOD

- **A METHOD OF PIECEWISE APPROXIMATION**

- **BY CONNECTING SIMPLE FUNCTIONS**

- **EACH VALID OVER A SMALL REGION / ELEMENT**

- **A PROCESS OF DISCRETIZATION**

ESSENTIAL STEPS IN FEM

- **DISCRETIZATION**
- **SELECTION OF THE DISPLACEMENT MODELS**
- **DERIVING ELEMENT STIFFNESS MATRICES**
- **ASSEMBLY OF OVERALL EQUATIONS / MATRICES**
- **SOLUTIONS FOR UNKNOWN DISPLACEMENTS**
- **COMPUTATIONS FOR THE STRAINS / STRESSES**

DISCRETIZATION (Fig 1.1)

- **SELECTING CERTAIN DISCRETE POINTS (NODES)**

- **FORMATION OF ELEMENT MESH**
  2D: 3/6 NODED TRIANGLES,
  QUADRILATERALS
  3D: TETRAHEDRAL, PRISMATIC etc

- **ELEMENTS INTERCONNECTED AT THE NODES**

- **DECIDE NUMBER, SIZE AND TYPE OF ELEMENT**

DISPLACEMENT MODELS (Fig 1.2)

- **IF NODAL DISPLACEMENTS ARE KNOWN**
- **DISPLACEMENT WITHIN IS COMPUTED**
- **USING SIMPLE FUNCTIONS (eg. POLYNOMIAL)**
- **INTRODUCES APPROXIMATION**
- **MODEL SHOULD SATISFY CERTAIN BASIC REQUIREMENTS TO MINIMIZE ERRORS**

DERIVATION OF THE ELEMENT MATRICES

- **EQUIVALENT FORCES AT THE NODES**
- **SPECIFY MATERIAL AND GEOMETRIC PROPERTIES**
- **STIFFNESS RELATES NODAL DISPLACEMENT TO FORCES**
- **DERIVE STIFFNESS MATRIX**
- **(MATRIX OF INFLUENCE COEFFICIENTS)**

DERIVATION OF OVERALL EQUATIONS / MATRICES

- **DISPLACEMENT AT A NODE TO BE SAME FOR ALL ADJACENT ELEMENTS**
- **COMBINE ELEMENT MATRICES**
- **DERIVE EXPRESSIONS FOR POTENTIAL ENERGY**
- $\prod = 1/2\ Q^T K Q - Q^T F$

SOLUTIONS FOR UNKNOWN DISPLACEMENTS

- **SPECIFY BOUNDARY CONDITIONS**
- **USE MINIMIZATION OF P.E. (say)**
- **DERIVE SIMULTANEOUS EQUATIONS**
- **KQ = F (Q's ARE UNKNOWNS)**
- **SOLVE USING NUMERICAL TECHNIQUES**
  1. **FOR LINEAR PROBLEMS: MATRIX AGEBRA TECHNIQUES**
  2. **FOR NON LINEAR PROBLEMS: MODIFY STIFFNESS / FORCE MATRIX AT EACH ITERATION**

COMPUTE STRESSES AND STRAINS

- **DERIVE STRAINS FROM DISPLACEMENTS**
- **DERIVE STRESSES FROM STRAINS**
- **USING SOLID MECHANICS PRINCIPLES**

FUNDAMENTALS OF MECHANICS (1D)

- **Stress Strain Relations**

  $$\varepsilon = du / dx$$

  $$\sigma_x = E\,\varepsilon = E\,du / dx$$

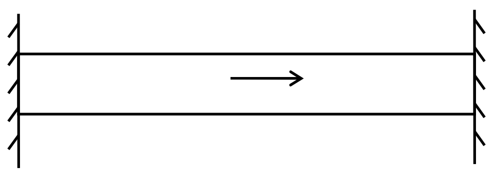
- **Force Equilibrium**

  $$d\sigma_x / dx + f = 0$$

  $$E\,d^2u / dx^2 + f = 0$$

SECOND ORDER DE TO BE SOLVED

BOUNDARY CONDITIONS



u = 0 at x = 0 and
u = 0 at x = L

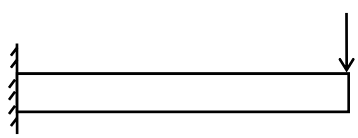
FOR BENDING PROBLEMS

- **EQUILIBRIUM EQUATION**
  $d^2M / dx^2 + q = 0$
  $\varepsilon = z\, d^2w / dx^2$
  $M = \sigma I / y$
- **FOURTH ORDER DE**

BOUNDARY CONDITIONS (in bending)

- **w, dw/dx, d^2w / dx^2 or d^3w / dx^3 AT THE BOUNDARY**

- **for instance**



$$w(0) = 0, \; dw / dx\,(0) = 0$$

A GENERAL 3D CASE

- **DEFORMATIONS**
  $\mathbf{u} = [u\ v\ w]^T$
- **STRESSES**
  $\sigma = [\sigma_x\ \sigma_y\ \sigma_z\ \tau_{yz}\ \tau_{xz}\ \tau_{xy}]^T$

- **STRAINS**
  $\varepsilon = [\varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \gamma_{yz}\ \gamma_{xz}\ \gamma_{xy}]^T$
  $= [\delta u/\delta x\ \ \delta v/\delta y\ \ \delta w/\delta z\ \ (\delta v/\delta z+\delta w/\delta y)\ ...]^T$

- **FORCES**
  BODY FORCES $[f_x\ f_y\ f_z]^T$
  TRACTIVE FORCES $[T_x\ T_y\ T_z]^T$
  POINT FORCES $[P_x\ P_y\ P_z]^T$

3D EQUILIBRIUM EQUATIONS

- **BODY FORCES (equilibrium of a volume element)**

$$\delta\sigma_x/\delta x + \delta\tau_{xy}/\delta y + \delta\tau_{xz}/\delta z + f_x = 0$$

$$\delta\tau_{xy}/\delta x + \delta\sigma_y/\delta y + \delta\tau_{yz}/\delta z + f_y = 0$$

$$\delta\tau_{xz}/\delta x + \delta\tau_{yz}/\delta y + \delta\sigma_z/\delta z + f_z = 0$$

- **TRACTIVE FORCES**

$$\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z = T_x$$
$$\tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z = T_y$$
$$\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = T_z$$

where $[n_x\ n_y\ n_z]^T$: surface normal

MATERIAL BEHAVIOR

- **LINEAR ISOTROPIC MATERIAL**
  (σ - ε relation defined using two constants)
  $$\varepsilon_x = (\sigma_x - \nu\, \sigma_y - \nu\, \sigma_z) / E$$
- **ORHOTROPIC (composites)**
  different properties in different directions
  upto nine constants to relate σ - ε
  For instance, composite materials
- **OTHER MATERIALS**
  non-linear isotropic (rubber)
  hypoelastic (incremental σ - ε relation)
  (geological materials)
  elasto-plastic (-do- with plasticity)
- **ONLY σ - ε relation changes**
- **FEM REMAINS SAME**

MINIMUM PE PRINCIPLE

- **BASIS OF FEM**
- $\prod = 1/2 \int \sigma^T \varepsilon \, dV - \int u^T f dV - \int u^T T dS - \sum u_i^T P_i$
- **AT EQUILIBRIUM** $\prod$ **IS A MINIMA**
- **FOR AN ASSUMED DISPLACEMENT FIELD**
- $\delta \prod / \delta a_i = 0$

ERRORS IN FEM

- **WRONG ASSUMTIONS**
- **USER ERRORS**
- **INAPPROPRIATE ELEMENT TYPE**
- **DISCRETIZATION ERRORS**
- **WRONG MESH SIZE**
- **YIELDING / BUCKLING OVERLOOKED**
- **WRONG SUPPORT CONDITIONS**
- **LARGE VARIATIONS IN STIFFNESSES**
- **PROGRAM BUGS + ROUNDING OFF**
- **IMPROPER TRAINING WITH SOFTWARE**

SOME POSSIBLE ANALYSIS TYPES

- **STATIC ANALYSIS**
- **DYNAMIC (MODAL / TRANSIENT)**
- **THERMAL / COMBINED STRESSES**
- **IMPACT STRESSES**
- **NON-LINEAR / PLASTIC MATERIALS**
- **COMPOSITE MATERIALS**
- **COMPLICATED LOADINGS AND BOUNDARY CONDITIONS**

TYPES OF APPLICATION AREAS

- **STRUCTURAL ENGINEERING APPLICATIONS**
- **HEAVY ENGINEERING COMPONENTS**
- **AUTOMOBILE PARTS**
- **AEROSPACE ENGINEERING**
- **NUCLEAR ENGINEERING**
- **TURBINE BLADES / OTHER POWER PLANT COMPONENTS**
- **AND MANY MORE**