ELSEVIER

# A new turbulence model for the axisymmetric plume

**Anupam Dewan,* Jaywant H. Arakeri, and J. Srinivasan**

*Department of Mechanical Engineering, Indian Institute of Science, Bangalore, India*

*A new turbulence model has been proposed for the axisymmetric plume. In this model the equations for the turbulent kinetic energy ($k$), its dissipation ($\epsilon$), temperature fluctuations ($\overline{t'^2}$), and intermittency ($\gamma$) have been solved. The equation for the dissipation ($\epsilon$) accounts for the effect of entrainment, which in turn requires the solution of a transport equation for the intermittency. The effect of buoyancy on Reynolds stress and radial heat flux has been modelled by the algebraic stress model (ASM). For the buoyancy production of turbulent kinetic energy a simple model (which assumes it to be proportional to $\sqrt{k\overline{t'^2}}$) has been used. A detailed comparison of the predictions has been made with measurements. The present $k-\epsilon-\overline{t'^2}-\gamma$ model is shown to simulate accurately the intermittency and the effects of buoyancy on mean and turbulent quantities.*
*© 1997 by Elsevier Science Inc.*

**Keywords:** buoyancy, intermittency, entrainment, $k-\epsilon$ model, algebraic stress model

## 1. Introduction

Most fluid flows in engineering applications and nature are turbulent. A plume is generated by a continuous source of buoyancy, which may be created by a source of heat or concentration. A plume is the simplest example of buoyancy-driven flows, and it has important applications in environmental studies, atmospheric science, and in many engineering applications, such as the cooling of electronic equipment and waste disposal in the atmosphere and oceans. The understanding of the behavior of the plume is important for modelling more complicated buoyancy-driven flows such as flames. In addition the plume provides a good example for illustrating the influence of buoyancy on turbulence.

Although the $k-\epsilon$ model is used widely to simulate nonbuoyant turbulent flows it has some limitations in the simulation of different free-shear flows. For example the model predicts correctly the growth rate of the plane jet, but it overpredicts that of the axisymmetric jet by 40%, termed the *plane jet/axisymmetric jet anomaly*.[1] Similarly the characteristics of the plane far-wake and the plane mixing layer are not predicted correctly by the $k-\epsilon$ model.[1]

Cho and Chung[1] proposed a modification to the dissipation equation to account for the effect of entrainment, and they solved a transport equation for the intermittency ($\gamma$) based on the Reynolds-averaged quantities. Ahn and Sung[2] showed that the $k-\epsilon-\gamma$ model is superior to the standard $k-\epsilon$ model for the plane momentumless wake. Recently Kim and Chung[3] showed that including the effect of entrainment in the Reynolds-stress transport model also improves the simulation of different free-shear flows. Two other modifications to the $k-\epsilon$ model have been proposed in the literature, which account for the effect of (1) vortex stretching[4] and (2) normal stresses[5] on the dissipation of turbulent kinetic energy. These modifications when used either independently or in combination (for example, as done by Cho and Chung[1]) lead to better predictions of the characteristics of simple nonbuoyant free-shear flows. However this is not true for buoyant free-shear flows (see Section 5).

In a plume the buoyancy affects both mean and turbulent quantities, and the buoyancy production of turbulent kinetic energy becomes important. Therefore plumes are more complex flows than nonbuoyant free-shear flows. Turbulence models of different complexities have been used to model the axisymmetric plume. These include the $k-\epsilon-\overline{t'^2}$ model,[6,7] the $k-W-\overline{t'^2}$ model ($W$ denotes mean-square vorticity fluctuations) proposed by Malin and Spalding,[8] and the more complex Reynolds-stress and turbulent-heat-flux transport model.[9] Two equation models require an additional transport equation for $\overline{t'^2}$ because temperature fluctuations are needed to compute the buoyancy production of turbulent kinetic energy.

Address reprint requests to Professor J. Srinivasan at the Indian Institute of Science, Department of Mechanical Engineering, Bangalore 560012, India.

*Present address: Department of Mechanical Engineering, Indian Institute of Technology, Guwahati 781 001, India.

Received 10 June 1996; revised 22 July 1997; accepted 29 August 1997.

Appl. Math. Modelling 1997, 21:709–719, November
© 1997 by Elsevier Science Inc.
655 Avenue of the Americas, New York, NY 10010

0307-904X/97/$17.00
PII S0307-904X(97)00096-6

The standard two-equation and Reynolds-stress models seriously overpredict the velocity and thermal growth rates of round jets and plumes unless empirical corrections are used to match the predictions with measurements (see, for example, Chen and Chen[7] and Malin and Spalding[8]). The objective of the present work is to improve the predictions of the axisymmetric plume using the $k-\epsilon-\overline{t'^2}$ model without tuning the model constants or employing empirical corrections.

In this paper we propose a new $k-\epsilon-\overline{t'^2}-\gamma$ model for the axisymmetric plume. This model combines the feature of the $k-\epsilon-\gamma$ model, i.e., the modification to account for the effect of entrainment on the rate of dissipation of turbulent kinetic energy, and that of the two-equation models for buoyancy-driven free-shear flows. The present model uses a simpler model for the buoyancy production of turbulent kinetic energy than the algebraic stress model (ASM) used in the literature. For the Reynolds shear stress and radial heat flux the ASM is adopted because it accounts for the effect of buoyancy on these correlations.

Section 2 of this paper provides the details of the present model and the numerical method used. In Section 3 the mean and turbulent quantities predicted by the present $k-\epsilon-\overline{t'^2}-\gamma$ model are compared with the $k-\epsilon-\overline{t'^2}$ model used by Chen and Chen[7] and the recent measurements of Shabbir and George[10] for the axisymmetric plume. Section 4 shows that including the effect of entrainment gives satisfactory predictions for different free-shear flows. Section 5 compares the effect of the three different modifications to the dissipation equation for the axisymmetric jet and plume.

## 2. Governing equations and turbulence model

The Reynolds-averaged boundary layer equations of continuity, axial momentum, and thermal energy for axisymmetric mean flow of an incompressible fluid in the cylindrical coordinate system with the Boussinesq approximation are:

Continuity:

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial (rv)}{\partial r}=0 \tag{1}$$

Axial momentum:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}[r(-\overline{u'v'})]+g\beta(T-T_\infty) \tag{2}$$

Thermal energy:

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}[r(-\overline{v't'})] \tag{3}$$

$u$ and $v$ are mean velocities in axial ($x$) and radial ($r$) directions. $T$ and $T_\infty$ are local mean and ambient temperatures, respectively. The last term on the right-hand side (RHS) of the momentum equation is the buoyancy term responsible for initiating fluid motion. The Reynolds stress in the momentum equation and the turbulent radial heat flux in the thermal energy equation are obtained from the following turbulence model (*Table 1* gives the important features of the present model).

Reynolds stress:

$$-\overline{u'v'}=\frac{1-c_0}{c_1}\frac{\overline{v'^2}}{k}\left[1+\frac{kg\beta(\partial T/\partial r)}{c_h\epsilon(\partial u/\partial r)}\right]\frac{k^2}{\epsilon}\frac{\partial u}{\partial r} \tag{4}$$

Radial heat flux:

$$-\overline{v't'}=\frac{1}{c_h}\frac{\overline{v'^2}}{k}\frac{k^2}{\epsilon}\frac{\partial T}{\partial r}\quad\text{where}\quad \overline{v'^2}=c_2k \tag{5}$$

Turbulent kinetic energy:

$$\frac{\partial k}{\partial t}+u\frac{\partial k}{\partial x}+v\frac{\partial k}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(c_k\frac{\overline{v'^2}k}{\epsilon}\right)\frac{\partial k}{\partial r}\right]+P_{k,s}+G_k-\epsilon \tag{6}$$

Rate of turbulent kinetic energy dissipation:

$$\frac{\partial \epsilon}{\partial t}+u\frac{\partial \epsilon}{\partial x}+v\frac{\partial \epsilon}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(c_\epsilon\frac{\overline{v'^2}k}{\epsilon}\right)\frac{\partial \epsilon}{\partial r}\right]+c_{\epsilon 1}\frac{\epsilon}{k}(P_{k,s}+G_k)+c_{\epsilon 4}\Gamma\frac{\epsilon^2}{k}-c_{\epsilon 2}\frac{\epsilon^2}{k} \tag{7}$$

$P_{k,s}=-\overline{u'v'}(\partial u/\partial r)$ and $G_k=g\beta\overline{u't'}$ are the shear and buoyancy productions of $k$, and the fourth term on the RHS of the dissipation equation represents the effect of entrainment on $\epsilon$, as explained in Section 2.2. The constants are given the following values: $c_0=0.55$, $c_1=2.2$, $c_h=3.2$, $c_2=0.53$, $c_k=0.225$, $c_\epsilon=0.15$, $c_{\epsilon 1}=1.43$, $c_{\epsilon 2}=1.92$, and $c_{\epsilon 4}=0.1$.[7] The following subsections discuss important features of the present model.

### 2.1. Models for turbulent transport quantities

The ASM is used for the Reynolds stress and the radial heat flux.[1] The ASM accounts for the effect of buoyancy on the Reynolds stress, which is not represented by the standard $k-\epsilon$ model. Chen and Nikitopoulos[6] and Chen and Chen[7] used an empirical correction for axisymmetric buoyant jets in which the Reynolds stress is multiplied by the factor $(1-0.465H)$ and the constant $c_{\epsilon 2}$ in the dissipation equation by the factor $(1-0.035H)$, where,

$$H=\left|\frac{\delta_u}{2u_0}\left[\frac{du_0}{dx}-\left|\frac{du_0}{dx}\right|\right]\right|^{0.2}.$$

**Table 1.** Comparison of the features of the present $k-\epsilon-\overline{t'^2}-\gamma$ model for the axisymmetric plume with the standard buoyancy-extended $k-\epsilon-\overline{t'^2}$ model

| Quantity | Original model (Chen and Chen[7]) | Present model |
|---|---|---|
| Reynolds shear stress | Algebraic Stress Model with empirical correction | ASM without empirical correction |
| Buoyancy production of turb. kinetic energy | Algebraic Stress Model | Model of Malin and Spalding[8] |
| Dissipation of turbulent kinetic energy | Equation with an empirical correction | (1) Empirical correction not used and (2) modified to consider the effect of entrainment |
| Dissipation of temperature fluctuations | By a constant time-scale ratio of velocity and temp. fluctuations (=1.25) | Different value of the constant (=1.79) |
| Intermittency | Not studied | Equation based on Reynolds-averaged quantities |

Here $u_0$ is the centerline axial velocity and $\delta_u$ is the velocity half width. In the axisymmetric plume $du_0/dx<0$, therefore $H>0$ and the effect of both the modifications is to reduce turbulence in the flow. The $k-W-\overline{t'^2}$ model also needs a similar correction for the $W$-equation,[8,11] but no empirical corrections are used in the present work.

Measurements for the axisymmetric plume show that the buoyancy production of $k$ $(=g\beta\overline{u't'})$ is about 30% of the shear production.[10] It is, therefore, important to model it accurately. The axial turbulent heat flux $(\overline{u't'})$ can be thought of as originating from parcels of fluid having a high temperature that will gain axial velocity due to buoyancy, i.e., we may expect $u'$ and $t'$ to be well correlated. It is clear that gradient transport is not at work and writing $-\overline{u't'}=(\nu_t/Pr_t)/(\partial T/\partial x)$ is inappropriate, where $Pr_t$ is the turbulent Prandtl number. Moreover $\partial T/\partial x$ is negligible in boundary-layer flows, and by the eddy viscosity relation $\overline{u't'}$ may change sign, but measurements show it to be positive. The ASM gives[7]

$$\overline{u't'}=\frac{k}{c_h\epsilon}\left[-\overline{u'v'}\frac{\partial T}{\partial r}-\overline{v't'}(1-c_{h1})\frac{\partial u}{\partial r}+g\beta(1-c_{h1})\overline{t'^2}\right]$$

with $c_h=3.2$ and $c_{h1}=0.5$ (8)

In Section 3 it will be shown that predictions with a simpler model $(\overline{u't'}=k_H\sqrt{k\overline{t'^2}})$ used by Malin and Spalding[8] are better than those that use the ASM. The value of the constant $k_H=0.56$ is recommended by them based on measurements of the correlation coefficient $(\overline{u't'}/\sqrt{\overline{u'^2}\,\overline{t'^2}})$ for the axisymmetric plume, which varies from 0.5–0.7 in different experiments. The model of Malin and Spalding[8] is not valid in general. For example if stratification is stable this model would predict a positive $\overline{u't'}$ whereas in reality there is a possibility of $\overline{u't'}$ changing sign.[11] To overcome this problem we suggest the following modification to Malin and Spalding's[8] model:

$$\left(\overline{u_i't'}=-\frac{\nu_t}{Pr_t}\frac{\partial T}{\partial x_i}+k_H\sqrt{k\overline{t'^2}}\right)$$

For a vertical plume in a uniform ambient the first term is negligible, and for a plume in a stably stratified medium the modified model permits negative $\overline{u't'}$. Moreover for homogeneous turbulence in a stably stratified medium the modified model permits negative values of the vertical turbulent heat flux. For this flow the modified model is similar to that proposed by Sommer and So,[12] who used a modified ASM since the standard ASM (equation [8]) failed to predict countergradient heat transport.

The temperature variance $\overline{t'^2}$ is determined from the following modelled transport equation[7]

$$\frac{\partial\overline{t'^2}}{\partial t}+u\frac{\partial\overline{t'^2}}{\partial x}+v\frac{\partial\overline{t'^2}}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(c_T\frac{k^2}{\epsilon}\right)\frac{\partial\overline{t'^2}}{\partial r}\right]+P_t-\epsilon_t \quad (9)$$

where $c_T=0.13$. $P_t=-2\overline{v't'}(\partial T/\partial r)$ and $\epsilon_t=\alpha\overline{(\partial t'/\partial x_j)(\partial t'/\partial x_j)}$ are the production and dissipation rate of $\overline{t'^2}$, respectively. $\epsilon_t$ is obtained from the relation $\epsilon_t=c_{T1}\epsilon\overline{t'^2}/k$. The constant $c_{T1}$ is the *time-scale ratio* of velocity and temperature fluctuations and is assumed to be constant across the width of the flow. Chen and Chen[7] used $c_{T1}=1.25$, but Malin and Younis[9] point out that for both plane and axisymmetric jets and plumes $c_{T1}=1.79$ produces better agreement with the measurements. Therefore the latter value is used in the present study.

### 2.2. *The dissipation equation*

The dissipation equation (7) contains one new term (fourth term on the RHS). This was introduced by Cho and Chung[1] in an attempt to have a "universal" model for all free-shear flows. $\Gamma$ $[=(k^{5/2}/\epsilon^2)[U_i/(U_kU_k)^{1/2}](\partial U_i/\partial x_j)(\partial U_i/\partial x_j)]$ is the *intermittency interaction invariant* and is a measure of the change in intermittency due to the entrainment of the outer irrotational fluid. The magnitude of $\Gamma$ depends on the rate of entrainment. The effect of $\Gamma$ on $\epsilon$ depends on the direction of movement of the turbulent-irrotational interface. If the entrainment is

by the inward movement of the outer irrotational fluid (as in a jet or in a plume), then $\gamma$ at a fixed location decreases and vice versa. A reduction in $\gamma$ means a reduction in turbulent length scale $l$ or an increase in $\epsilon$ as $l \simeq k^{3/2}/\epsilon$. For thin axisymmetric flows $\Gamma = (k^{5/2}/\epsilon^2)\partial u/\partial r(\partial\gamma/\partial r)$,[1,2] and so the intermittency distribution ($\gamma$) is needed, which is obtained by solving a transport equation (described in Section 2.3).

### 2.3. *Transport equation for intermittency*

Libby[13] was the first to propose a model for a guessed transport equation for the intermittency. Dopazo[14] derived an exact transport equation for intermittency by conditioning the instantaneous continuity equation with an intermittency indicator function. Intermittency for the axisymmetric plume was obtained by solving the following modelled transport equation:

$$\frac{\partial\gamma}{\partial t} + u\frac{\partial\gamma}{\partial x} + v\frac{\partial\gamma}{\partial r} = D_g + S_g \tag{10}$$

$D_g$ represents the transport of $\gamma$ due to the mean velocity difference between the turbulent and irrotational fluids and is effective only in the intermittent region. The following diffusion model for $D_g$ is used[1]:

$$D_g = \frac{1}{r}\frac{\partial}{\partial r}\left[r(1-\gamma)\frac{\nu_t}{\sigma_g}\frac{\partial\gamma}{\partial r}\right], \quad \text{with } \sigma_g = 1.0.$$

$S_g$ represents the conversion rate of the outer irrotational fluid into the turbulent fluid and involves the geometry of the interface. The following model for $S_g$ is used, with $c_{g1} = 1.6$, $c_{gb1} = 0.25$, $c_{g2} = 0.15$, and $c_{g3} = 0.16$:

$$S_g = c_{g1}\gamma(1-\gamma)\frac{P_{k,s}+P_{k,n}}{k} + c_{gb1}\gamma(1-\gamma)\frac{G_k}{k} + c_{g2}\frac{k^2}{\epsilon}\frac{\partial\gamma}{\partial x_j}\frac{\partial\gamma}{\partial x_j} - c_{g3}\gamma(1-\gamma)\frac{\epsilon}{k}\Gamma \tag{11}$$

Cho and Chung[1] proposed this model by modifying the original model of Byggstoyl and Kollmann.[15,16] In the literature the original intermittency transport equation has been used with the *conditional $k-\epsilon$ and Reynolds-stress transport models*. In conditional models different transport equations for all mean and turbulent quantities are solved in the turbulent and nonturbulent zones. This makes conditional models much more complex compared to the $k-\epsilon-\gamma$ model. The second term on the RHS of equation (11) involving $G_k$ is new and is introduced to represent the effect of buoyancy; its effect is discussed in Section 3.

**Boundary conditions.** —Since the flow is assumed to be *axisymmetric*, computations were performed for one azimuthal location. The *zero flux* boundary condition on the centerline for all the variables was used. At the edge of the computational domain $u$, $k$, $\epsilon$, $\overline{t'^2}$, and $\gamma$ were specified equal to zero and $T$ was taken to be equal to the ambient value. At the discharge point top hat profiles of all the variables were specified. Computations were stopped when the *self-similarity* was reached. Self-similarity was assumed to be reached when the changes in the growth rates and centerline values of mean and turbulent quantities were within 1%.

**Numerical method and code validation.** —The finite volume method[17] was used to solve the system of equations. The effect of the axial step size and the number of radial grids was studied.[18] All results reported in the following sections are with an axial step size of 5% of the local velocity half width ($\delta_u$) and 125 nonuniform radial grids.

Two test cases were selected to validate the code, namely, the axisymmetric jet using the $k-\epsilon-\gamma$ model of Cho and Chung[1] and the axisymmetric plume using the $k-\epsilon-\overline{t'^2}$ model of Malin.[19] The comparison with these two test cases permits validation of all the modifications to the $k-\epsilon$ model employed in the present study. For both the axisymmetric jet and plume the mean and turbulent quantities were found to be within 2% of those reported by Cho and Chung[1] and Malin.[19] For the axisymmetric jet the computations were made with the discharge conditions such that height nondimensionalised by the Morton length scale $L_M$ is much less than 1.[10]

## 3. Comparison of predictions with measurements

Many measurements for the axisymmetric plume have been reported in the literature.[10,20–26] Chen and Rodi[27] have provided a review of measurements up to 1980. Shabbir and George[10] have highlighted the sensitivity of the boundary conditions on the measurements and have shown that a small ambient stratification or use of small screens to prevent ambient disturbances from affecting the main flow could result in large errors. They showed that the difference between their measurements and others was not due to different instruments used or due to measurements not being made at large downstream locations, but was mainly because of different boundary conditions. Hence we have used the experimental data of Shabbir and George[10] for comparison.

*Table 2* shows important mean and turbulent quantities predicted by the present $k-\epsilon-\overline{t'^2}-\gamma$ model, the original $k-\epsilon-\overline{t'^2}$ model (used by Chen and Chen,[7] without the empirical correction to the model constants, Section 2.1), the Reynolds-stress and heat flux transport (RSHFT) model,[9] and the measurements by Shabbir and George[10] ($B_u$ and $B_t$ in *Table 2* are decay constants for centerline velocity and temperature). *Table 2* shows that all mean and turbulent quantities predicted by the present model are in better agreement with the measurements than the other two models.

Figures 1–13 compare the mean and turbulent quantities predicted by the present model and original model with the measurements. Results for the Reynolds-stress and heat flux transport model[9] are not shown as they are similar to those of the $k-\epsilon-\overline{t'^2}$ model. The curve fits given by Shabbir and George[10] to their measurements

**Table 2.** Predicted mean and turbulent quantities for the self-similar axisymmetric plume compared with measurements

| Quantity | $B_u$* | $B_t$† | $\frac{d\delta_u}{dx}$ | $\frac{d\delta_t}{dx}$ | $\frac{\overline{u'v'}_{max}}{u_0^2}$ | $\frac{\overline{v't'}_{max}}{u_0 \Delta T_0}$ | $\frac{\overline{u't'}_{max}}{u_0 \Delta T_0}$ | $\frac{\sqrt{\overline{t'^2}_0}}{\Delta T_0}$ |
|---|---|---|---|---|---|---|---|---|
| Present $k-\epsilon-\overline{t'^2}-\gamma$ model | 3.64 | 9.53 | 0.103 | 0.097 | 0.023 | 0.034 | 0.062 | 0.41 |
| Original $k-\epsilon-\overline{t'^2}$ model without empirical correction (Chen and Chen[7]) | 2.94 | 6.37 | 0.151 | 0.137 | 0.032 | 0.047 | 0.081 | 0.86 |
| Reynolds stress and H.F. transport model (Malin and Younis[9]) | — | — | 0.143 | 0.150 | 0.029 | 0.050 | 0.088 | 0.65 |
| Measurements (Shabbir and George[10]) | 3.40 | 9.40 | 0.107 | 0.100 | 0.026 | 0.029 | 0.070 | 0.39 |

*$B_u = (\pi/4)^{1/3} u_0 x^{1/3} B_0^{-1/3}$ (Chen and Rodi[27]).
†$B_t = (\pi/4)^{2/3} g\beta \Delta T_0 x^{5/3} B_0^{2/3}$.

have been used for comparison. For some quantities the curve fits do not accurately represent the measured values over some region of the plume. Therefore conclusion on the performance of the models are based on the overall comparison of all quantities. All quantities in *Figures 1–13* have been nondimensionalised by the local mean quantities ($u_0$, $\Delta T_0$, and $\delta_u$).

### 3.1. Mean quantities

The profiles of the mean quantities predicted by the present model are superior to those of the original model, which predicts flatter profiles close to the outer edge of the plume (*Figures 1* and *2*). The present model predicts correctly both the profile shape and magnitude of all the terms of mean momentum and buoyancy equations, including the buoyancy term in the momentum equation (*Figures 3* and *4*). The buoyancy term is responsible for the observed larger velocity gradients in the plume than in the jet.

### 3.2. Turbulent quantities

The profiles of $\overline{u'v'}$ and $\overline{v't'}$ predicted by the present model are in good agreement with the measurements (*Figures 5* and *6*). However the predictions by the $k-\epsilon$

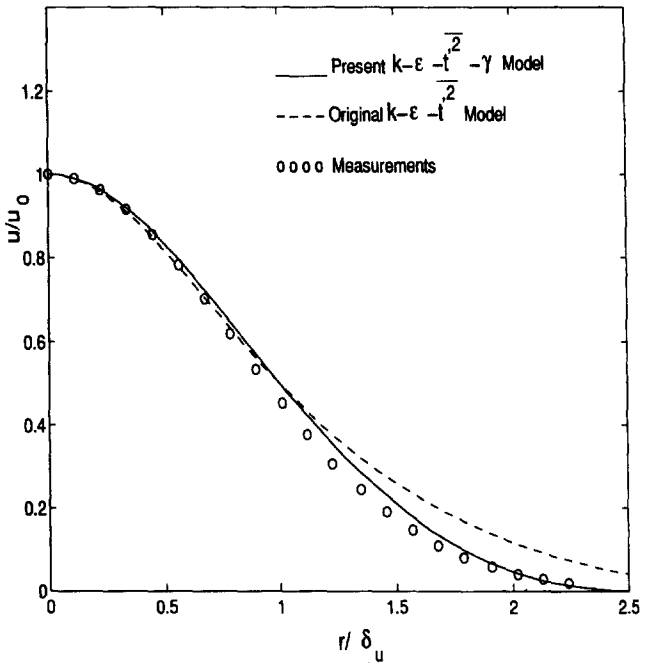



**Figure 1.** Predicted mean velocity profiles for the axisymmetric plume compared with the measurements.

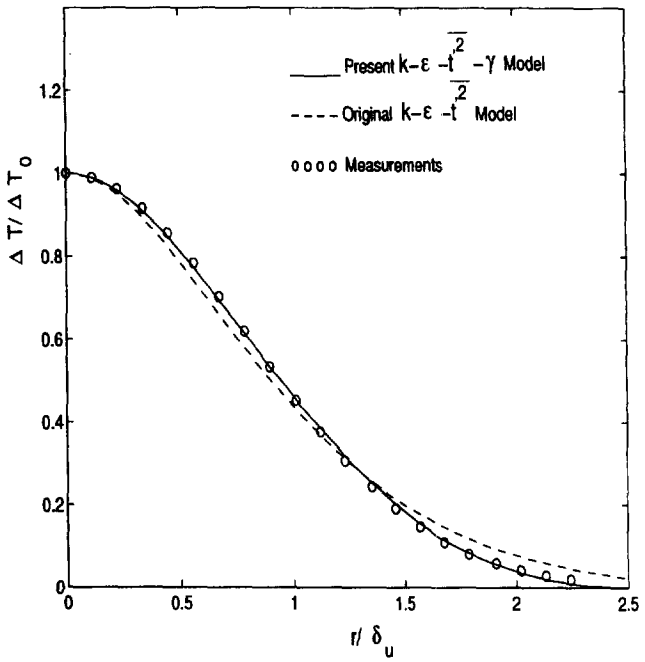



**Figure 2.** Predicted mean temperature profiles for the axisymmetric plume compared with the measurements.

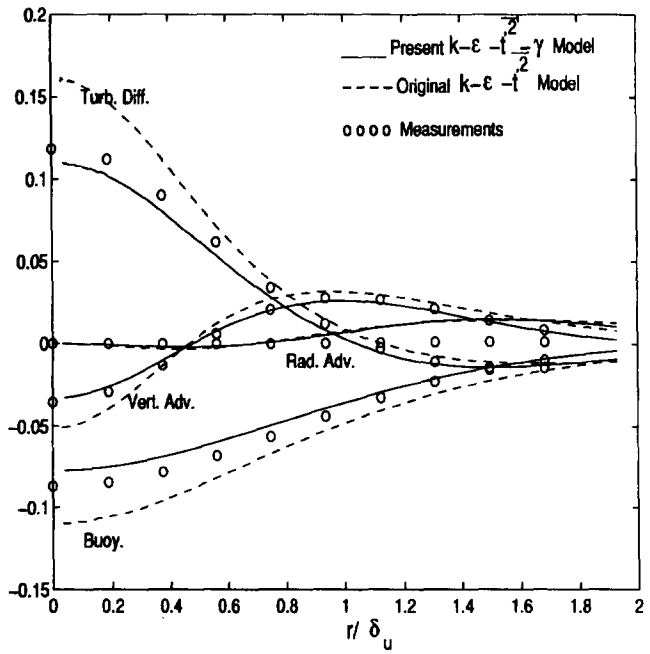



**Figure 3.** Predicted terms of the mean momentum equation for the axisymmetric plume compared with the measurements.

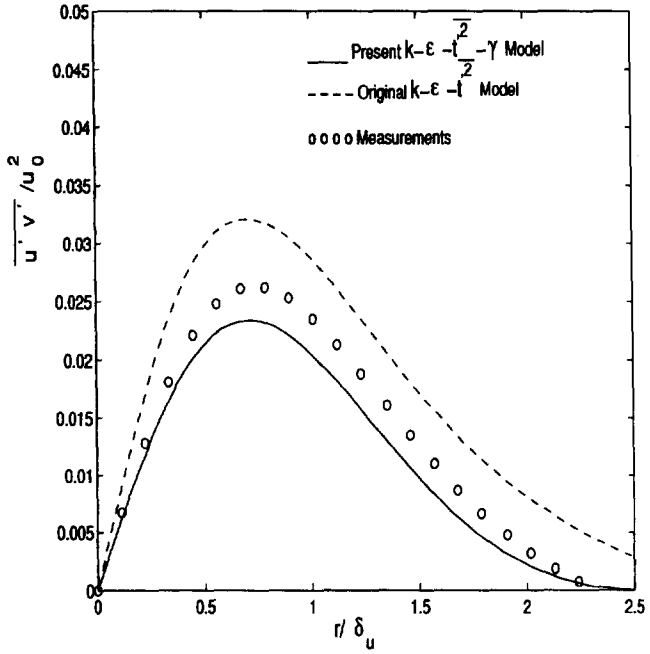



**Figure 5.** Predicted Reynolds shear stress profiles for the axisymmetric plume compared with the measurements.

$-\overline{t'^2}$ model are about 30% larger than the measurements. The prediction of the model of Malin and Spalding[8] for the axial heat flux agrees better with the measurements than the predictions using the ASM (*Figure 7*). Since the buoyancy production ($G_k$) is about 30% of the shear production ($P_{k,s}$) the difference between the two models for $\overline{u't'}$ does not significantly affect the predictions of the other quantities. The profiles of the Reynolds stress and radial heat flux using the full transport model[9] are similar to that shown here using the ASM. This shows that the simpler ASM is adequate to represent the effects of buoyancy on Reynolds stress and radial heat flux. However for $\overline{u't'}$ even the full transport model does not predict the correct profile shape.[9] The use of the more general nonequilibrium form of the ASM for $\overline{u't'}$ ($P_{k,s} + G_k \neq \epsilon$ and $P_t \neq \epsilon_t$) does not lead to any significant change in profile shape.[19]

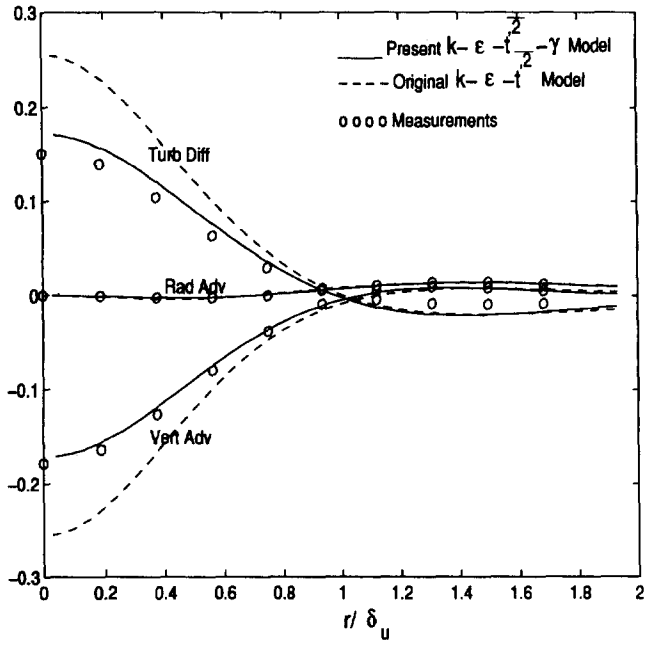



**Figure 4.** Predicted terms of the mean temperature equation for the axisymmetric plume compared with the measurements.

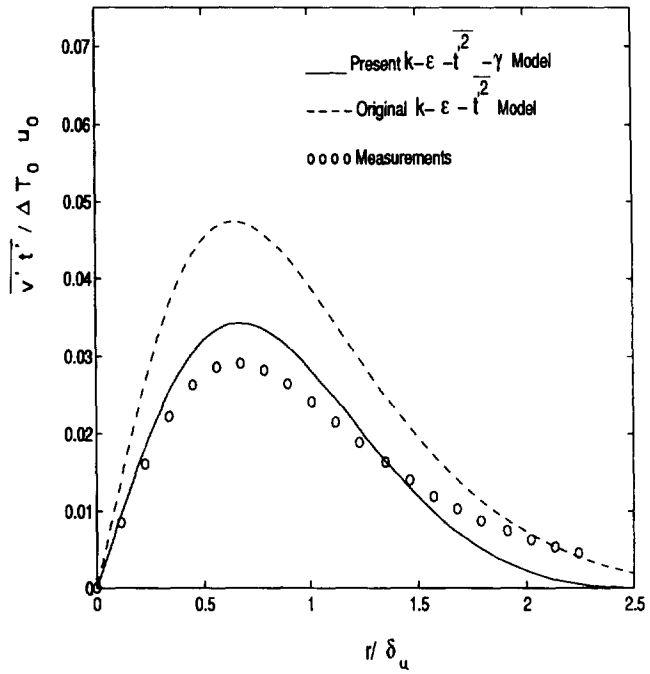



**Figure 6.** Predicted radial turbulent heat flux profiles for the axisymmetric plume compared with the measurements.

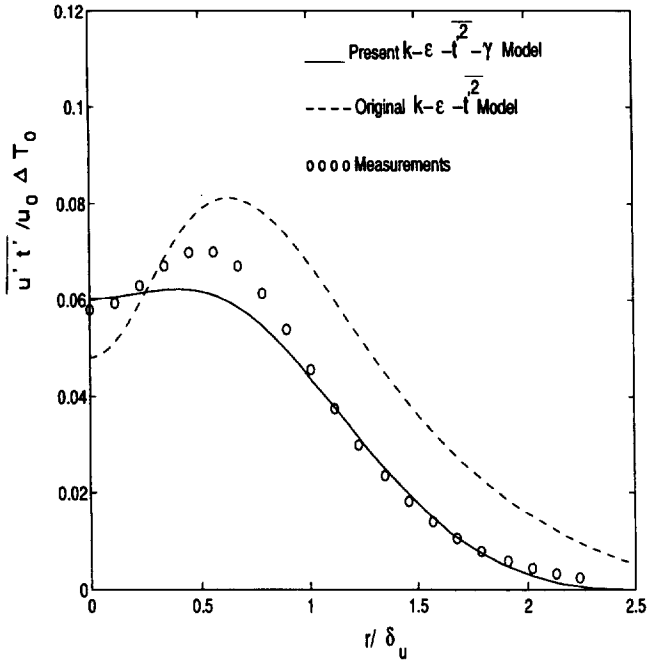



**Figure 7.** Predicted axial heat flux profiles for the axisymmetric plume compared with the measurements.

It is surprising that the prediction of the turbulent kinetic energy profiles by the present model are not in good agreement with the measurements, especially close to centerline, in contrast to the predictions by the original model (*Figure 8*). This is probably due to the inability of the curve fit given by Shabbir and George[10] to represent accurately the trend of axial velocity fluctuations close to the centerline. All the terms of the $k$ equation predicted by the present model are in good agreement with the measurements (*Figure 9*), except turbulent diffusion. Even the predicted radial distribution of the dissipation compares well with measurements. Close to the axis turbulent diffusion predicted by the model is positive, whereas, the measured value is zero. Shabbir and George[10] mention that their measurements of third moments (which include turbulent diffusion) may have large errors. The measurements of third moments also show scatter of almost 100%. In the outer region, however, there is reasonable agreement between the prediction of the diffusion and the measurements. In contrast all the terms of the $k$ equation predicted by the original model are larger than the measured values.

The predictions of $\overline{t'^2}$ by the $k-\epsilon-\overline{t'^2}$ model are about 100% larger than the measurements over the whole width of the plume (*Figure 10*). The predictions of temperature fluctuations by the present model are, however, in better agreement with the measurements. The difference between $\overline{t'^2}$ predicted by the two models is primarily due to the different values of the constant $c_{T1}$ used in the two models (*Table 1*). Moreover the predictions of $k$ and $\epsilon$ by the present model also contribute to the better predictions of $\overline{t'^2}$ (through the modelling of $\epsilon_t$). The observations indicate that as the buoyancy increases the temperature fluctuations increase by as much as 100%.[23] The present model is able to reproduce this behavior.

The different terms in the $\overline{t'^2}$ equation predicted by the $k-\epsilon-\overline{t'^2}$ model are about three to four times larger than the measurements (*Figure 11*). In contrast the predictions of all the terms, including dissipation, by the present

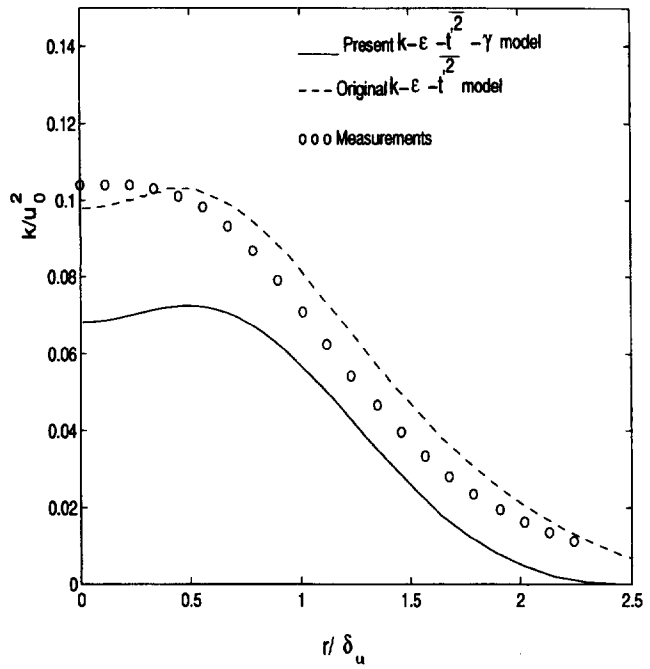



**Figure 8.** Predicted turbulent kinetic energy profiles for the axisymmetric plume compared with the measurements.

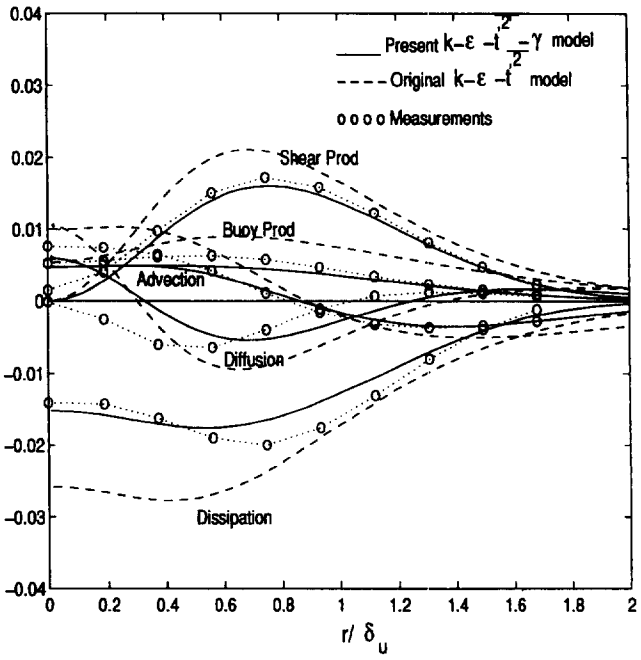



**Figure 9.** Predicted terms of the turbulent kinetic energy equation for the axisymmetric plume compared with the measurements.

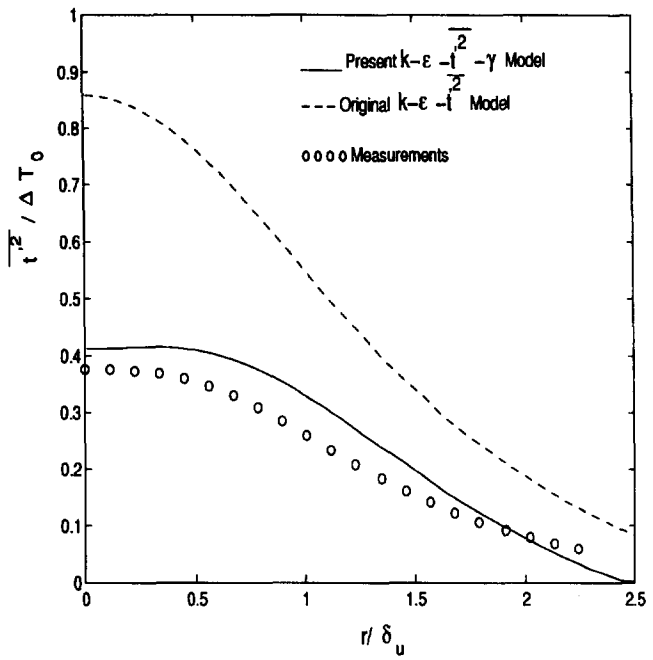



**Figure 10.** Predicted temperature fluctuations profiles for the axisymmetric plume compared with the measurements.

model are in better agreement with the observations. Shabbir and George[10] have indicated that their measurements of all the terms of the $\overline{t'^2}$ equation were more accurate than that of the $k$ equation. *Figure 11* shows that the modelling of the dissipation of temperature fluctuations using the constant time-scale ratio of velocity and temperature fluctuations (= 1.79) is adequate and that a transport equation for $\epsilon_t$ is not needed.

### *3.3. Intermittency and entrainment*

Intermittency is an important turbulent quantity that provides insight regarding the turbulence structure and the entrainment process. The intermittency profile for the axisymmetric plume predicted by the present model agrees well with the measurements by Papanicolaou and List[23] (*Figure 12*). The entrainment coefficient ($\alpha$) predicted by the present model is 0.097, and the measured value is 0.108. $\alpha$ predicted by the $k-\epsilon-\overline{t'^2}$ model is 0.171, i.e., 58% larger than the measurements. Measurements for the axisymmetric jet show $\alpha = 0.053 - 0.055$.[24] ($\alpha$ is a measure of the downstream increase in the volume of the flow and has been defined in the Nomenclature.) The present model predicts that the rate of entrainment is higher for the axisymmetric plume than for the axisymmetric jet, in agreement with the experimental observations of Papanicolaou and List.[24] They showed that in the plume large eddies entrain unmixed fluid that reach the axis and therefore lead to higher entrainment compared to that for the jet. Good predictions of the intermittency, the entrainment, and the rate of dissipation of $k$ show that the effect of entrainment on the rate of dissipation considered in the present study is appropriate for the axisymmetric plume.

### *3.4. Present modification to the intermittency equation*

It can be shown that with an increase in the value of the constant $c_{gb1}$ the deviation between the predicted and the measured intermittency increases, but the turbulent quan-

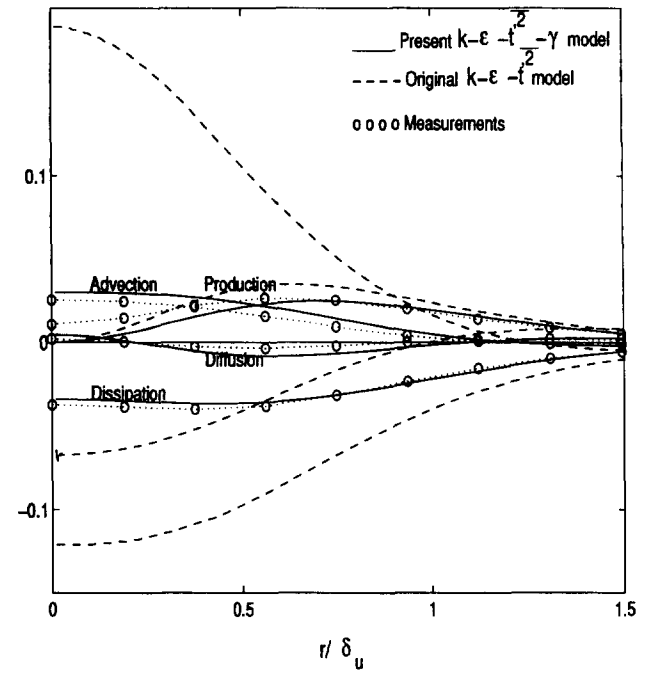



**Figure 11.** Predicted terms of the temperature fluctuations equation for the axisymmetric plume compared with the measurements.

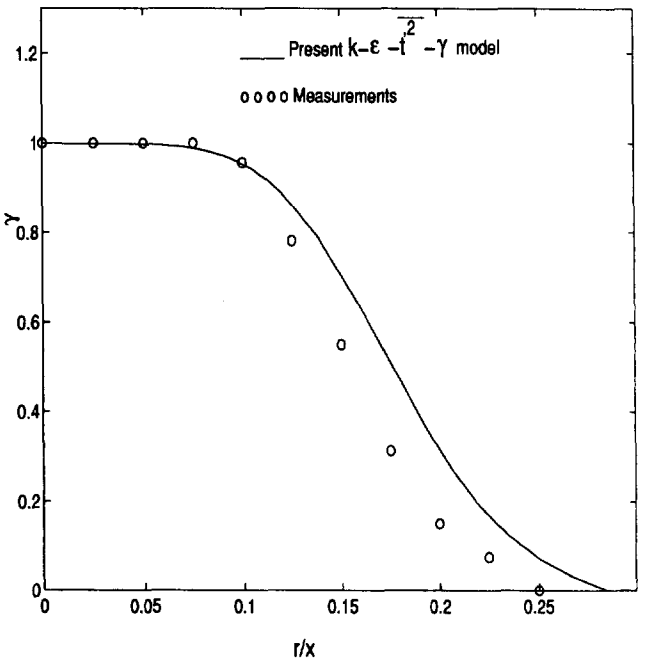



**Figure 12.** Predicted intermittency for the axisymmetric plume compared with the measurements.

tities and the velocity growth rate approach measured values ($d\delta_u/dx$ for $c_{gb1} = 0.00$ and 0.25 are 0.099 and 0.103, respectively). All the results presented in this section are with $c_{gb1} = 0.25$. Note that the changes in $d\delta_u/dx$ of about 5% that occur on account of varying the value of the constant $c_{gb1}$ are much smaller compared to the change of about 50% that occurs on account of the inclusion of the intermittency interaction invariant ($\Gamma$) in the $\epsilon$ equation.

## 4. Application of the $k-\epsilon-\gamma$ model to different free-shear flows

*Table 3* shows that the $k-\epsilon-\gamma$ model improves the predictions of different free-shear flows. It may be noted that for the computations using the $k-\epsilon-\gamma$ model only one modification to the dissipation equation, i.e., to account for the effect of entrainment, has been incorporated. The $k-\epsilon$ model overpredicts the growth rates of the jet and plume, it underpredicts that of the wakes and mixing layer (*Table 3*), and the $k-\epsilon-\gamma$ model removes all these anomalies. The $k-\epsilon-\gamma$ model should be tested for other free-shear flows, i.e., plane plume, plane jet, and radial jet. The data in *Table 3* have been taken from Pope,[4] Hanjalic and Launder,[5] Patel and Scheuerer,[28] Cho and Chung,[1] and Ahn and Sung[2] (the widths $l_0$, $S$, and $L$ and the velocity ratio $R$ have been defined in the Nomenclature). Cho and Chung[1] used a model for eddy viscosity that is different from the standard model and that accounts for the effect of the outer irrotational flow. It can be shown that for the axisymmetric jet and plume this model predicts the velocity growth rate, which is about 5% larger than that predicted by the standard model. In the present computations for the planar flows using the $k-\epsilon-\gamma$ model (shown in *Table 3*) the effect of outer irrotational flow on the eddy viscosity has been assumed to be negligible.

## 5. Anomalous effect of the three modifications for jet and plume

As mentioned in the Introduction apart from the entrainment term two additional correction terms to the dissipation equation have been proposed in the literature to remove the limitations of the $k-\epsilon$ model. One is the vortex stretching term $[c_{\epsilon 3}X(\epsilon^2/k)]$, which is nonzero only for axisymmetric flows and was introduced by Pope[4] to remove the plane jet/axisymmetric jet anomaly. The other term proposed by Hanjalic and Launder[5] $[c_{\epsilon 1}(\epsilon/k)3P_{k,n}]$ sensitises the $\epsilon$ equation to normal stresses. $P_{k,n} = -(\overline{u'^2} - \overline{v'^2})(\partial u/\partial x)$ is the production of $k$ due to the normal stresses and is negligible compared to $P_{k,s}$ and $G_k$. Similarly the contribution of the normal stresses to the momentum equation is negligible. However for consistency this term needs to be retained in these two equations without the factor 3.[1] Following Hanjalic and Launder[5] $(\overline{u'^2} - \overline{v'^2}) = c_{uv}k$ with $c_{uv} = 0.33$ and $c_{\epsilon 3} = 0.79$ as prescribed by Pope.[4] These two terms were also included by Cho and Chung[1] and were found to improve the predictions of the axisymmetric jet.

Although both of the above terms are as dominant in the plume as in the jet they have adverse effect on the axisymmetric plume characteristics. The predictions with the normal stress term alone show an increase in the velocity and thermal growth rates of the plume by about 150 and 125%, respectively, compared to the measured values,[10] and the vortex stretching term alone increases the growth rates by 75 and 70%, respectively. For the axisymmetric jet the predicted velocity growth rates with the inclusion of the three modifications independently, namely, for vortex stretching, normal stresses and entrainment, are 0.088, 0.098, and 0.098, respectively, and the measured value is 0.086.[1] The effect of the three modifications for the jet and the plume can be explained from their contributions to the $\epsilon$-equation (*Figure 13*). For both the flows the standard source term (sum of the second, third, and fifth terms on the RHS of equation [7]) is much smaller in the outer region compared to the region close

**Table 3.** Comparison of the predicted growths of different free-shear flows by the $k-\epsilon$ and $k-\epsilon-\gamma$ models

| Flow | Quantity | Measurement | By the standard $k-\epsilon$ model | By the $k-\epsilon-\gamma$ model |
|---|---|---|---|---|
| Axisymmetric jet | $d\delta_u/dx$ | 0.086 | 0.125 | 0.098 |
| Plane far-wake | $\partial S/\partial x$ | 0.098 | 0.060 | 0.088 |
| Plane momentumless wake | $l_0/d$ at $x/d = 40, 60$* | 1.12, 1.35 | 0.80, 0.90 | 1.11, 1.27 |
| Plane mixing layer | $dL/dx$ (for $R = 0$)† | 0.115 | 0.094 | 0.106 |
| Axisymmetric plume | $d\delta_u/dx$ | 0.107 | 0.151‡ | 0.103§ |

*Growth rate is nonlinear.
†For $R > 0$ the deviation between measurements and predictions by two models decreases.
‡By the buoyancy extended $k-\epsilon-\overline{t'^2}$ model without the empirical correction.[7]
§By the present $k-\epsilon-\overline{t'^2}-\gamma$ model.

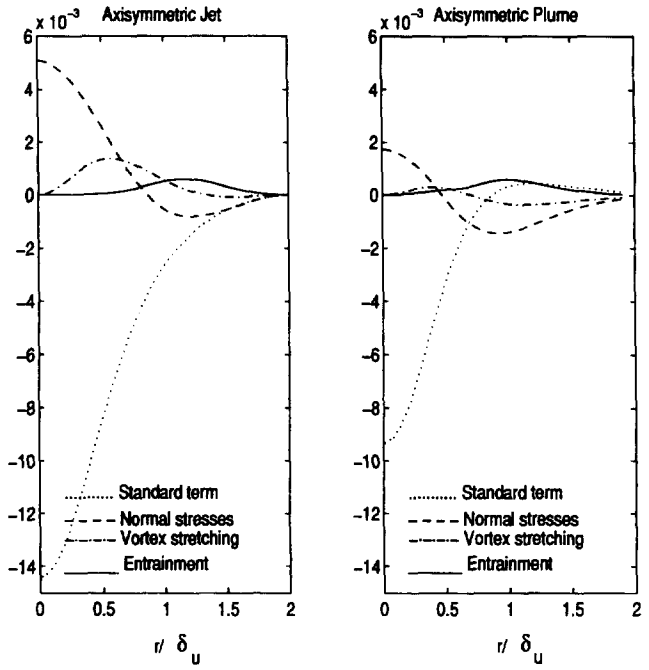



**Figure 13.** Comparison of the source terms of the dissipation equation for the axisymmetric jet and the axisymmetric plume.

to the centerline (for the plume the standard term decays faster compared to the jet). For $r/\delta_u > 1$ the terms corresponding to the three modifications are as large as the standard term itself. Therefore the percentage of change in the rate of dissipation due to small modifications in the $\epsilon$-equation is large. Two other flows with adverse effects due to the modification for the vortex stretching are the radial jet[29] and the axisymmetric jet in a coflowing stream.[1] In a buoyant jet the effects of buoyancy gradually become important as the height nondimensionalised by the Morton length scale $L_M$ increases. It is clear from the above that if we study a buoyant jet using the $k-\epsilon-\overline{t'^2}$ model and consider the three modifications to the dissipation equation independently, only the modification accounting for the effect of entrainment (incorporated in the present model) would give reasonable predictions.

## 6. Concluding remarks

It has been shown that the inclusion of the effect of entrainment on the rate of dissipation of turbulent kinetic energy in the buoyancy-extended $k-\epsilon-\overline{t'^2}$ turbulence model significantly improves the prediction of the axisymmetric plume characteristics. No empirical correction is required to match the measurements. The predictions of intermittency by a modelled transport equation and entrainment compare well with the measurements for the axisymmetric plume. The $k-\epsilon-\gamma$ model has been shown to be the superior to the standard $k-\epsilon$ model for a variety of free-shear flows.

In contrast to the earlier models, in which the ASM was used to determine the production of turbulent kinetic energy due to buoyancy ($g\beta\overline{u't'}$), a simpler model (which assumes $\overline{u't'}$ to be proportional to $\sqrt{k\overline{t'^2}}$) is shown to improve its prediction.

Predictions of all mean and turbulent quantities of the axisymmetric plume using the present $k-\epsilon-\overline{t'^2}-\gamma$ model are better than any other previously used model, i.e., the $k-\epsilon-\overline{t'^2}$, $k-W-\overline{t'^2}$, and Reynolds-stress and heat flux transport models.

## Acknowledgments

The first author (AD) thanks the Council of Scientific and Industrial Research, Government of India, for financial support through the award of Research Associateship. The authors thank the referees for comments and suggestions that led to an improvement in the quality of the paper.

## Nomenclature

| | |
|---|---|
| $B_0$ | discharge buoyancy flux = $2\pi\int_0^{D/2} ug(\Delta\rho/\rho)r\,dr$ |
| $B_u, B_t$ | decay constant for centerline velocity, temperature of plume |
| $c', c'_g$ | model constants in turbulence model |
| $D$ | jet discharge diameter |
| $D_g, S_g$ | source terms in intermittency equation |
| $g$ | gravitational acceleration |
| $G_k$ | buoyancy production of $k$ ($=g\beta\overline{u't'}$) |
| $H$ | empirical correction for axisymmetric plume |
| $k$ | turbulent kinetic energy |
| $k_H$ | constant in model for buoyancy production of $k$ |
| $l$ | turbulent length scale |
| $l_0$ | width of momentumless wake corresponding to minimum velocity |
| $L$ | width of the plane mixing layer (distance between points with $u=\sqrt{0.9}(U_H-U_L)+U_L$ and $u=\sqrt{0.1}(U_H-U_L)+U_L$) |
| $L_M$ | Morton length scale = $M_0^{3/4}/B_0^{1/2}$ |
| $m$ | local volume flux = $\int_0^\infty u2\pi r\,dr$ |
| $M_0$ | discharge momentum flux = $2\pi\int_0^{r_0} u^2 r\,dr$ |
| $P_{k,s}, P_{k,n}$ | production of $k$ by shear, normal stresses |
| $P_t$ | production of temperature fluctuations |
| $Pr_t$ | turbulent Prandtl number |
| $r$ | radial coordinate |
| $R$ | velocity ratio of plane mixing-layer ($=U_L/U_H$) |
| $S$ | width of plane far wake ($=0.5u_\infty\delta_u/u_d$) |
| $T$ | mean temperature |
| $\Delta T$ | $T-T_\infty$ |
| $\overline{t'^2}$ | mean square temperature fluctuations |
| $u, v$ | streamwise, radial mean velocity |
| $u_d$ | velocity defect of wake |
| $\overline{u't'}, \overline{v't'}$ | axial, normal turbulent heat flux |
| $\overline{u'v'}$ | Reynolds shear stress |
| $\overline{v'^2}$ | normal velocity fluctuations |

| | |
|---|---|
| $U_H, U_L$ | higher, lower free-stream velocity in plane mixing-layer |
| $W$ | mean square vorticity fluctuations |
| $x$ | streamwise coordinate; location from virtual origin of plume |
| $X$ | mean vortex stretching invariant |

*Greek symbols*

| | |
|---|---|
| $\alpha$ | entrainment coefficient $= \dfrac{dm/dx}{2\pi\delta_{u,e}u_0}$ |
| $\beta$ | coefficient of volumetric expansion |
| $\delta_u, \delta_{u,e}$ | velocity half width, exponential width |
| $\epsilon, \epsilon_t$ | rate of dissipation of turbulent kinetic energy, temperature fluctuations |
| $\gamma$ | intermittency |
| $\Gamma$ | intermittency interaction invariant |
| $\nu_t$ | eddy viscosity |

*Subscripts*

| | |
|---|---|
| 0 | centerline value |
| $\infty$ | ambient value |
| *dis* | discharge quantities |
| *max* | maximum |

## References


1. Cho, J. R. and Chung, M. K. A $k-\epsilon-\gamma$ equation turbulence model. *J. Fluid Mech.* 1992, **237**, 301–322
2. Ahn, J. W. and Sung, H. J. Prediction of two-dimensional momentumless wake by $k-\epsilon-\gamma$ model. *AIAA J.* 1995, **33(4)**, 611–617
3. Kim, S. K. and Chung, M. K. Roles of pressure transport and intermittency for computation of turbulent free shear flows. *Int. J. Heat Fluid Flow* 1995, **16(3)**, 194–201
4. Pope, S. B. An explanation of the turbulent round jet/plane jet anomaly. *AIAA J.* 1978, **16(3)**, 279–281
5. Hanjalic, K. and Launder, B. E. Sensitizing the dissipation equation to irrotational strains. *ASME J. Fluids Eng.* 1980, **102**, 34–40
6. Chen, C. J. and Nikitopoulos, C. P. On the near field characteristics of axisymmetric turbulent buoyant jets in a uniform environment. *Int. J. Heat Mass Trans.* 1979, **22**, 245–255
7. Chen, C. J. and Chen, C. H. On prediction and unified correlation for decay of vertical buoyant jets. *ASME J. Heat Trans.* 1979, **101**, 532–537
8. Malin, M. R. and Spalding, D. B. The prediction of turbulent jets and plumes by use of the $k-W$ model of turbulence. *PCH Phys. Chem. Hydrodynam.* 1984, **5(2)**, 153–198
9. Malin, M. R. and Younis, B. A. Calculation of turbulent buoyant plumes with a Reynolds stress and heat flux transport closure. *Int. J. Heat Mass Trans.* 1990, **33(10)**, 2247–2264
10. Shabbir, A. and George, W. K. Experiments on a round turbulent buoyant plume. *J. Fluid Mech.* 1994, **275**, 1–32
11. Malin, M. R. The decay of mean and turbulent quantities in vertical forced plumes. *Appl. Math. Modelling* 1987, **11**, 301–314
12. Sommer, T. P. and So, R. M. C. On the modelling of homogeneous turbulence in a stably stratified flow. *Phys. Fluids* 1995, **7**, 2766–2777
13. Libby, P. A. On the prediction of intermittent turbulent flows. *J. Fluid Mech.* 1975, **68**, 273–295
14. Dopazo, C. On conditioned averages for intermittent turbulent flows. *J. Fluid Mech.* 1977, **81**, 433–439
15. Byggstoyl, S. and Kollmann, W. Closure model for intermittent turbulent flows. *Int. J. Heat Mass Trans.* 1981, **24(11)**, 1811–1822
16. Byggstoyl, S. and Kollmann, W. A closure model for conditioned stress equations and its application to turbulent shear flows. *Phys. Fluids* 1986, **29**, 1430–1440
17. Patankar, S. V. *Numerical Heat Transfer and Fluid Flow*. Hemisphere Publishing Corporation, New York, 1980
18. Dewan, A. Turbulence modelling of thick axisymmetric wall-bounded flows and axisymmetric plume. Ph.D. Thesis, Department of Mechanical Engineering, Indian Institute of Science, Bangalore, India, 1996
19. Malin, M. R. The prediction of vertical turbulent plumes by use of the $k-\epsilon$ and $k-W$ models of turbulence. Report No. 9, Department of Mechanical Engineering, Imperial College of Science and Technology, London, UK, 1983
20. Nakagome, H. and Hirata, M. The structure of turbulent diffusion in an axisymmetrical thermal plume. *Proceedings of the 1976 ICHMT Seminar on Turbulent Buoyant Convection*, 1977, pp. 361–372
21. George, W. K., Alpert, R. L., and Tamanini, F. Turbulence measurements in an axisymmetric buoyant plume. *Int. J. Heat Mass Trans.* 1977, **20**, 1145–1154
22. Beuther, P. D., Capp, S. P., and George, W. K. Momentum and temperature balance measurements in an axisymmetric turbulent plume. *ASME publication 79-HT-42*, 1979
23. Papanicolaou, P. N. and List, E. J. Statistical and spectral properties of tracer concentration in round buoyant jets. *Int. J. Heat Mass Trans.* 1987, **30(10)**, 2059–2071
24. Papanicolaou, P. N. and List, E. J. Investigations of round vertical turbulent buoyant jets. *J. Fluid Mech.* 1988, **195**, 341–391
25. Dai, Z., Tseng, L. K., and Faeth, G. M. Structure of round, fully developed, buoyant turbulent plumes. *ASME J. Heat Trans.* 1994, **116**, 409–417
26. Dai, Z., Tseng, L. K., and Faeth, G. M. Velocity statistics of round, fully developed, buoyant turbulent plumes. *ASME J. Heat Trans.* 1995, **117**, 138–145
27. Chen, C. J. and Rodi, W. *Vertical Turbulent Buoyant Jets*. Pergamon, London, 1981
28. Patel, V. C. and Scheuerer, G. Calculation of two-dimensional near and far wakes. *AIAA J.* 1982, **20**, 900–907
29. Rubel, A. On the vortex stretching modification of the $k-\epsilon$ turbulence model: Radial jets. *AIAA J.* 1985, **23(7)**, 1129–1130