

The Basel Problem is the problem of finding out the sum $1 + 1/2^2 + 1/3^2 + \dots$. It turns out that this sum is equal to $\pi^2/6$. In the following, we will see a proof of this fact using the Fourier Series. We will need to use some Hilbert Space terminologies and techniques which will be developed as needed.

First, let's introduce the periodic function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(x) = x$ when $x \in [-\pi, \pi)$, with a period of 2π , which means that $\forall x \in \mathbb{C}, f(x + 2\pi) = f(x)$. Thus by defining the values of $f(x)$ for $x \in [-\pi, \pi)$, we have defined its values everywhere.

Second, we need to look at the Fourier series representation of $f(x)$ when $x \in [-\pi, \pi)$. The reason why will become apparent when we reach the solution, but for now let's just go with it. The Fourier Series is actually a Fourier basis representation of periodic functions in the Hilbert Space $L^2[-\pi, \pi)$. The $L^2[-\pi, \pi)$ is **complete inner product** vector space whose elements are **square integrable** functions over the interval $[-\pi, \pi)$.

A function is square integrable in $[-\pi, \pi)$ if $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$. The integral is the Lebesgue integral, which is why the space has L in its name. All continuous functions over $[-\pi, \pi)$ are square integrable. An inner product vector space is a vector space equipped with an inner product. An inner product over a vector space V is a function $p : V \times V \rightarrow \mathbb{C}$ with the following properties.

Property 1 Conjugate symmetry: $p(x, y) = \overline{p(y, x)}$

Property 2 Linearity in the first argument: $p(ax + bz, y) = ap(x, y) + bp(z, y)$

Property 3 Positive definiteness: if $x \neq 0$ then $p(x, x) > 0$

In the above, $x, y, z \in V$, and $a, b \in \mathbb{C}$. Further the line above $p(y, x)$ signifies the complex conjugate; for a complex number $c = p + iq$, its complex conjugate is $\bar{c} = p - iq$. And, we also have $|c|^2 = c\bar{c}$.

An inner product in a vector space induces a norm on the vector space, and the norm in turn induces a metric on the vector space. The existence of a metric on a vector space means that we can talk about limits of (Cauchy) sequences of elements within the vector space. The limit of such sequences may or may not exist within the vector space itself. When the limit lies within the vector space itself, the space is called complete.

The inner product on the space $L^2[-\pi, \pi)$ is $p(f, g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$. We will represent this with the more familiar notation $\langle f, g \rangle \equiv p(f, g)$ as it is more convenient and less confusing.

The periodic function $f(x) = x$ defined above is in fact in the space $L^2[-\pi, \pi)$. Therefore, just like any other vector in a vector space, it can be represented as a linear combination of a complete orthonormal basis of the vector space in consideration. The particular orthonormal basis that we will use is what gives rise to the Fourier Series. While the proof that the basis that we are about to discuss in a moment is in fact a complete orthonormal basis that spans the $L^2[-\pi, \pi)$ space is beautiful and interesting in its own right, however we shall not discuss it here. These proofs may be found in any standard functional analysis textbook.

The basis that we are interested in is made up of the functions $e_n = \frac{1}{\sqrt{2\pi}}e^{inx}$ for $n \in \mathbb{Z}$. Do not confuse the e in e_n which is the standard notation of basis vectors with the e in e^{inx} which is the Euler's number ≈ 2.718 . Notice that the basis is countably infinite, implying that the dimension of the vector space $L^2[-\pi, \pi)$ is ∞ in some sense. Regardless, just like in any other vector space, we can write an element of the vector space as a unique linear combination of the basis vectors. For any vector space V that has a complete orthonormal basis $(e_j)_{j \in J}$, and for any element $x \in V$, we can write x as

$$x = \sum_{j \in J} p(x, e_j)e_j,$$

where $p(x, e_j)$ is the inner product in the vector space V .

Similarly, in $L^2[-\pi, \pi)$, we can write any function f as

$$f(x) = \sum_{n=-\infty}^{n=\infty} \langle f, e_n \rangle e_n.$$

Also, in any other vector space, we know that the square of the norm of the vector $\|x\|^2 = \sum_{j \in J} p(x, e_j)^2$. Take for example a vector $x = (1, 2, 3)^\top$ in $\mathbb{R}^3$, we know that $\|x\|^2 = 1^2 + 2^2 + 3^2$. Similarly in $L^2[-\pi, \pi]$ the square of the L^2 -norm of a function (defined as $\langle f, f \rangle$) is the same as $\|f\|^2 = \sum_{n=-\infty}^{\infty} |\langle f, e_n \rangle|^2$.

Remark. These are just analogies, without rigorous definitions or proofs. There are more rigorous definitions and proofs to be found in standard textbooks, but their discussion will make this document too long and will postpone the discussion of the main subject of this document for longer than we want.

Now, we are ready to compute the Fourier series representation of $f(x) = x$. For doing so, we need to compute $\langle f, e_n \rangle \equiv \hat{f}(n)$. Let's compute these coefficients.

$$\begin{aligned}\hat{f}(n) &= \langle f, e_n \rangle \\ &= \int_{-\pi}^{\pi} f(x) \overline{e_n} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2\pi}} e^{inx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-inx} dx\end{aligned}$$

We will need to use the method of integration by parts to solve this. Recall that $(uv)' = u'v + uv'$ which implies the integration by parts rule that $uv = \int u dv + \int v du$. We shall take $u = x$, and $dv = e^{-inx} dx$. We will also need $du = dx$, and $v = \int dv = \int e^{-inx} dx$. Further, assume $y = e^{-inx}$ to get $dy = -ine^{-inx} dx$. Which implies that $\int dy = \int -ine^{-inx} dx$, or $\int e^{-inx} dx = \frac{y}{-in} = \frac{e^{-inx}}{-in}$.

Now we have u, du, v, dv . We just need $\int v du$, which is $\int \frac{e^{-inx}}{-in} dx = \frac{e^{-inx}}{(-in)^2}$. This gives us the following ($\hat{f}(n) = \int u dv = uv - \int v du$).

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \left[\frac{x e^{-inx}}{-in} - \frac{e^{-inx}}{-n^2} \right]_{-\pi}^{\pi}$$

Further simplifying this gives

$$\sqrt{2\pi} \hat{f}(n) = \left[\frac{i x e^{-inx}}{n} + \frac{e^{-inx}}{n^2} \right]_{-\pi}^{\pi}$$

Let's analyze these term by term. First let's assign them names for easy manipulation. Let $T_1 = \frac{i x e^{-inx}}{n}$, and $T_2 = \frac{e^{-inx}}{n^2}$. Further, note that both T_1 and T_2 are functions of x that will only be evaluated at $x = -\pi$ or $x = \pi$. We will need to make use of Euler's identity which states $e^{i\theta} = \cos \theta + i \sin \theta$. And since $\cos(-\theta) = \cos \theta$ and $\sin(-\theta) = -\sin \theta$, we further have $e^{-i\theta} = \cos \theta - i \sin \theta$.

Let's first analyze T_1 .

$$\begin{aligned}T_1(x) &= \frac{i}{n} x (\cos nx - i \sin nx) \\ T_1(\pi) &= \frac{i}{n} \pi (\cos n\pi - i \sin n\pi) \\ &= \frac{i}{n} \pi (-1)^n \\ T_1(-\pi) &= -\frac{i}{n} \pi (-1)^n\end{aligned}$$

Now, let's analyze T_2 .

$$\begin{aligned}T_2(x) &= \frac{1}{n^2} \cos nx - \sin nx \\T_2(\pi) &= \frac{1}{n^2} (-1)^n \\T_2(-\pi) &= \frac{1}{n^2} (-1)^n\end{aligned}$$

Now, we can finally solve for $\hat{f}(n)$ as follows.

$$\begin{aligned}\sqrt{2\pi} \hat{f}(n) &= \frac{i}{n} \pi (-1)^n + \frac{1}{n^2} (-1)^n - \left(-\frac{i}{n} \pi (-1)^n + \frac{1}{n^2} (-1)^n \right) \\&= \frac{i2\pi(-1)^n}{n} \\ \implies \hat{f}(n) &= \frac{i\sqrt{2\pi}(-1)^n}{n}\end{aligned}$$

Remark. The above is only applicable for when $n \neq 0$. For $n = 0$, it is clear that $\hat{f}(n) = 0$ because the integral is $\int_{-\pi}^{\pi} x dx = 0$ due to symmetry. Further, $f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ can be simplified to find the result in closed form. $\hat{f}(n) e_n = i\sqrt{2\pi}(-1)^n/n \cdot e^{inx}/(\sqrt{2\pi}) = i(-1)^n/n(\cos nx + i \sin nx) = i(-1)^n \cos nx/n + (-1)^{n+1} \sin nx/n$. Now observe that $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$, and that there will be a $-n$ term for every n term. This means that all the $\cos$ terms will cancel each other out, leaving only $\sin$ terms. The final form looks like this $f(x) = \lim_{N \rightarrow \infty} \sum_{n=1}^N 2(-1)^{n+1} \sin nx/n$.

The last step is to compute the square of the norm of the function $f(x) = x$ in $L^2[-\pi, \pi]$ in two different ways, one using the inner product, and the other using the coefficients of the “vector”. Let's first compute the norm by “the inner product” way.

$$\begin{aligned}\langle f, f \rangle &= \int_{-\pi}^{\pi} f(x) \overline{f(x)} dx \\&= \int_{-\pi}^{\pi} x^2 dx \\&= \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} \\&= \frac{2\pi^3}{3}\end{aligned}$$

Next, let's compute this “the sum of squares of the coefficients” way.

$$\begin{aligned}\|f\|^2 &= \sum_{n=-\infty}^{n=\infty} |\hat{f}(n)|^2 \\&= \sum_{n=-\infty}^{n=\infty} \left| \frac{i\sqrt{2\pi}(-1)^n}{n} \right|^2 \\&= \sum_{n=-\infty}^{n=\infty} \frac{2\pi}{n^2}\end{aligned}$$

We need to make one further adjustment, to this. From the symmetry of square, we know that $\frac{2\pi}{n^2} = \frac{2\pi}{(-n)^2}$, and we can change the lower limit from $-\infty$ to 1 to get the following.

$$\|f\|^2 = \sum_{n=1}^{\infty} \frac{4\pi}{n^2}$$

We can now equate the two.

$$\begin{aligned} \langle f, f \rangle &= \frac{2\pi^3}{3} = \|f\|^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2} \\ \implies \frac{2\pi^3}{3} \cdot \frac{1}{4\pi} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ \implies \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

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A single line python code using numpy verifies this. The line of code is

```
pi = np.sqrt(6 * np.sum(np.square(1/np.arange(1, n))))
```

and you can set n to any value you like in the above code. When we set $n = 50,000,000$, we get the following output.

```
Approximation of pi with n = 50,000,000 is 3.1415926344911984
Error is 1.91e-08 (actual np.pi = 3.141592653589793)
```

If you have the patience to compute 50 million squares, their reciprocals, and maintain a sum, then enjoy computing π this way! Happy New Year!