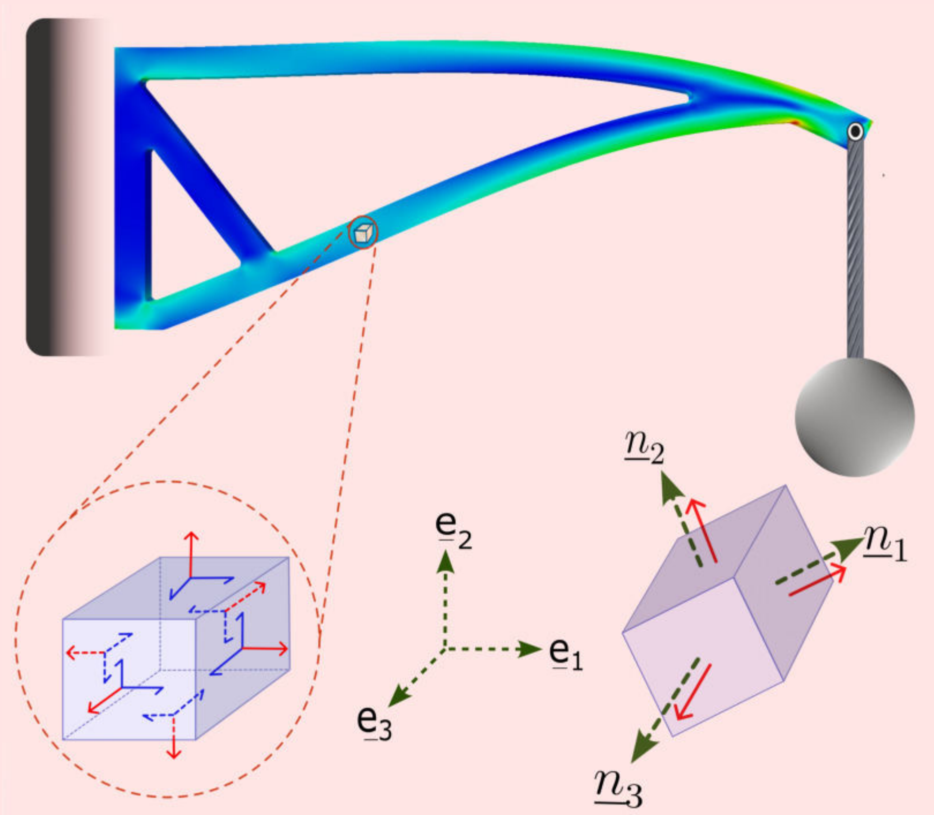
Solid Mechanics for Undergraduates

using vectors and tensors



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Solid Mechanics for Undergraduates

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To
my parents who inspire me to work tirelessly

Ajeet Kumar

Contents

Preface

This book is intended for readers who wish to thoroughly understand the basic concepts required for venturing into the mechanics of deformation in solids. The topics covered in the book are taught at undergraduate level in courses often named as *Mechanics of Materials*, *Strength of Materials* etc. to students of Mechanical, Civil, Applied Mechanics and Materials engineering disciplines. However, the emphasis in such courses tend to be more on application rather than understanding the involved concepts. Most of the books available in the market are also designed that way. However, for students who desire to pursue higher studies or sit in competitive exams after their undergraduate education, they will appreciate the style of the present book. A great emphasis has been placed here on deriving the theoretical concepts with full mathematical rigor and in further illustrating their application through several solved and unsolved examples. The book will also be useful for Masters and PhD students working in the broad area of Solid Mechanics and for teachers of undergraduate colleges in solidifying their grip over the subject. The book relies heavily on the use of vectors and tensors in order to make derivations concise and big mathematical expressions compact which further help in elucidating the underlying physics better. Students in second year of their college education may find it difficult to work with tensors. Accordingly, the first chapter of the book has been carefully crafted to elucidate the various concepts involving tensors and also introduce the mathematical language used in the book. Each chapter in the book ends with long subjective-type solved examples to illustrate the concepts covered and how to further use them in solving engineering problems. Likewise, several objective-type questions are also given at the end of each chapter to help the readers in preparing for competitive exams. Web-links to my NPTEL video lectures for every theory section of the book are also given in the HTML version of the book (hosted on my IIT webpage) in case a reader desires in learning through video.

The various topics covered in the book are discussed according to the following outline. Chapter 1 introduces the concepts of vectors, tensors and various mathematical operations involving them. Chapter 2 introduces the concepts of traction vector, stress tensor, stress matrix and its transformation. In chapter 3, equilibrium equations of linear elasticity are derived along with a discussion on boundary conditions. Chapter 4 comprehensively discusses the various concepts involving stress tensor such as *principal stress components, Mohr's circle, stress invariants, stress decomposition* etc. In chapter 5, the concept of strain is introduced and further generalized for arbitrary deformation in three-dimensional solids. The mathematical formulas for various types of strains, e.g., *longitudinal strain, shear strain and volumetric strain* are also derived here along with a discussion on strain compatibility conditions. Chapter 6 discusses linear stress-strain relation in three-dimensional elastic solids for general anisotropic materials. The discussion is later focused on isotropic materials. In chapter 7, cylindrical coordinate system is introduced and the different components of stress and strain tensors in this coordinate system are also obtained. The equations of elasticity are also rederived in this coordinate system. These derivations are later used in solving various axisymmetric deformation problems such as extension, torsion and inflation of solid and hollow cylinders. Chapter 8 discusses comprehensively bending of beams - both uniform and non-uniform as occurring in beams having symmetrical as well as unsymmetrical cross-sections. The concept of shear center is also discussed later on and its formula is derived for thin and open cross-sections. In chapter 9, beam theory is introduced for obtaining the transverse deflection of slender structures. The *Euler-Bernouli beam theory* and the *Timoshenko beam theory* are both derived here. Buckling of beams is then discussed. In chapter 10, *energy methods* is introduced and various concepts related to it are discussed in detail. The chapter ends with illustrating how this method can be used to solve complex beam deflection problems which otherwise cannot be solved using Euler-Bernouli or Timoshenko beam theory in a conventional way. Finally, chapter 11 briefly discusses various failure theories.

The book is a result of several years of my teaching at IIT Delhi. My own knowledge of the subject increased manifold after rich interaction with students of IIT Delhi. I would like to express my heartfelt gratitude to my teacher Prof. S.K. Roy Chowdhury who germinated in me a liking for Solid Mechanics while I took the course *Mechanics of Materials* under him during my third semester at IIT Kharagpur. I would also like to thank NPTEL which provided me with a platform to prepare and disseminate a video lecture series on this subject - this immensely helped me in refining my own concepts and in thinking of alternate and simpler ways of explaining several topics. I would like to acknowledge the book *Advanced Mechanics of Solids* by Prof. L.S. Srinath - some example problems in the present book are taken from there although their solutions have been prepared independently by me. I would also like to thank Prof. Rajdip Nayek with whom I co-taught this course once and we then prepared solution to several of the questions discussed in this book. Finally, I would like to gratefully acknowledge my students Roushan Kumar, Siddhant Jain and Mohit Garg who have immensely helped me in drafting this book.

It is also my endeavor to continually improve this book. I would be really happy to receive any feedback or comment on this book from readers.

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Chapter 1

Mathematical Preliminaries

In this introductory chapter, we will get familiar with vectors, tensors and various mathematical operations involving them. We will learn about dot-product, cross-product and tensor product of two vectors, single-contraction and double contraction of two tensors, how to take their gradient and divergence etc. The concepts covered here will be used throughout the book. Hence, they are of paramount importance.

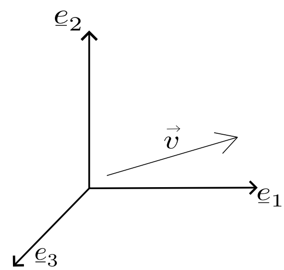


Figure 1.1: A vector $\overrightarrow{v}$ drawn in a Cartesian coordinate system

1.1 A vector and its representation

In layman's term, a vector has both magnitude and direction. It is denoted by an arrow as shown in Figure 1.1. The length of the arrow signifies the vector's magnitude while the arrow's orientation signifies the vector's direction.[1] We have also shown a Cartesian coordinate system there whose basis vectors are $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$. The component of a vector $\overrightarrow{v}$ along the basis vector $\underline{e}_i$ is given by

$$v_i = \overrightarrow{v} \cdot \underline{e}_i. \tag{1.1}$$

[1]Generally speaking, vectors can be moved parallelly in space without changing them. One exception is the position vector whose one end is tied to a point, usually the origin of the coordinate system.

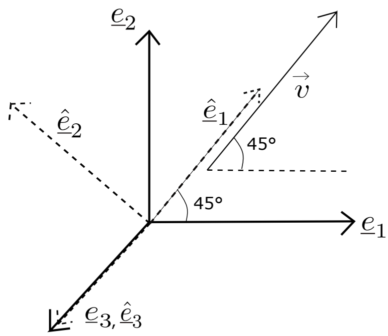


Figure 1.2: A vector $\vec{v}$ being observed from two different coordinate systems

Geometrically, this implies projection of the vector on to the basis vector $\underline{e}_i$. The three components of a vector can be written together in a column form which we denote by the symbol $\left[\vec{v}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)}$ and write

$$\left[\vec{v}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{1.2}$$

The subscript in $\left[\vec{v}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)}$ signifies the coordinate system relative to which the vector components have been obtained. At this point, one should note that $\left[\vec{v}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)}$ and $\vec{v}$ are not the same: the former is the representation of the latter in the specified coordinate system. More importantly, a vector is independent of the coordinate system but its representation changes from one coordinate system to the other. To elaborate this point, think of a unit vector $\vec{v}$ lying in space and being viewed from two different coordinate systems having basis vectors $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ and $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$ respectively - see Figure 1.2. The dashed coordinate system there can be obtained by rotating the other coordinate system by 45° about $\underline{e}_3$ axis. The vector $\vec{v}$ itself lies in $\underline{e}_1$ - $\underline{e}_2$ plane and makes an angle of 45° from $\underline{e}_1$ axis. This also implies that $\vec{v}$ is directed along $\hat{\underline{e}}_1$, i.e., the first basis vector of the dashed coordinate system. The representation of $\vec{v}$ in the two coordinate systems will thus be

$$\left[\vec{v}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \left[\vec{v}\right]_{(\hat{\underline{e}}_1,\hat{\underline{e}}_2,\hat{\underline{e}}_3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{1.3}$$

Thus, the representation (column form) of a vector varies from one coordinate system to the other but the vector itself is still directed in the same way in space and hence is independent of the coordinate system. It is also useful to note the below form for a vector:

$$\begin{aligned} \vec{v} &= \sum_{i=1}^{3} (\vec{v} \cdot \underline{e}_i)\, \underline{e}_i = \sum_{i=1}^{3} v_i\, \underline{e}_i \\ &= \sum_{i=1}^{3} (\vec{v} \cdot \hat{\underline{e}}_i)\, \hat{\underline{e}}_i = \sum_{i=1}^{3} \hat{v}_i\, \hat{\underline{e}}_i. \end{aligned} \tag{1.4}$$

From now on, we will denote a vector $\vec{v}$ by $\underline{v}$, i.e, instead of the overhead arrow, we will use an underbar.

1.2 Dot-product, Cross-product and Tensor-product of two vectors

Let us now talk about three ways in which two vectors can be operated together. We begin first with dot product. The dot product of two vectors yields a scalar quantity and hence it is also called the *scalar product*. It is defined as follows:

$$\begin{aligned} \underline{a}\cdot\underline{b} &= \sum_{i=1}^{3} a_i b_i = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}_{1\times 3} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times 1} = \left[\underline{a}\right]^T \left[\underline{b}\right] \\ &= ||\underline{a}||\ ||\underline{b}||\ \cos(\theta). \end{aligned} \tag{1.5}$$

Basically, the dot product of two vectors is obtained by summing the product of the corresponding components of those two vectors. From this definition, it appears that the dot product will be different in different coordinate systems since it involves the components of two vectors. However, as per the second formula above which is geometric in nature, it also equals the product of the magnitude of the two vectors multiplied by cosine of the angle between them. The second definition implies that the dot product of two vectors is independent of the coordinate system since neither the magnitude of a vector nor does the angle between two vectors change upon change of coordinate system.

The cross product of two vectors yields a vector due to which it is also called the *vector product*. Like

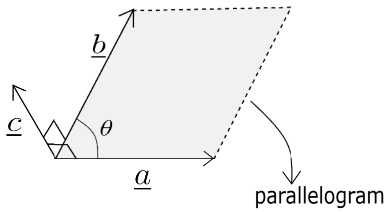


Figure 1.3: Graphical illustration of cross-product $\underline{a}\times\underline{b}$: area of the shaded parallelogram is the magnitude of cross-product whereas the plane normal $\underline{c}$ is along the cross-product direction

dot product, it is also independent of the coordinate system. Geometrically, it is defined as follows:

$$\underline{a}\times\underline{b} = ||\underline{a}||\ ||\underline{b}||\ \sin(\theta)\ \underline{c}. \tag{1.6}$$

Here $\underline{c}$ is a unit vector perpendicular to the plane formed by $\underline{a}$ and $\underline{b}$. Its direction is given by the right hand thumb rule: when we curl our fingers from $\underline{a}$ towards $\underline{b}$, the thumb points towards $\underline{c}$. The cross-product also equals the area vector of a parallelogram whose sides are formed by $\underline{a}$ and $\underline{b}$ (see Figure 1.3 for a graphical illustration). In a coordinate system, say $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$, the cross product can be written as follows:

$$\left[\underline{a}\times\underline{b}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \left[\underline{a}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} \times \left[\underline{b}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} (a_2b_3 - a_3b_2) \\ (a_3b_1 - a_1b_3) \\ (a_1b_2 - a_2b_1) \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{1.7}$$

Thus, the cross product of two vectors can also be realized as the product of a skew symmetric matrix (a matrix whose diagonal elements are 0 and off-diagonal elements are negative of each other) times the column of the second vector. The components of the skew-symmetric matrix are formed by the components of the first vector $\underline{a}$. In order to easily remember how to form the skew-symmetric matrix from the components of $\underline{a}$, one can think of the following trick: to get the component in the i^{th} row and j^{th} column, the component of $\underline{a}$ that will be used will be the third index (other than i and j). For example, for 1^{st} row and 2^{nd} column of the matrix, 3^{rd} component a_3 will be used. One then just has to remember where to place the negative signs. We also say

$$[\underline{a}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = axial \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{1.8}$$

Basically, whenever we have a skew symmetric matrix, we can form a column from the three independent entries of that matrix and call the resultant column the axial/axial vector of that skew symmetric matrix.

Let us now talk of tensor product. This is a different kind of product which we may not have heard of yet. Through this product, we will also introduce a general notion of tensors. The tensor product of two vectors yields what is called a second order tensor which we write as follows:

$$\underline{a} \otimes \underline{b} = \underline{\underline{C}}. \tag{1.9}$$

Here $\underline{\underline{C}}$ (with two underbars) denotes a second order tensor. The tensor product is represented as follows in a coordinate system:

$$[\underline{\underline{C}}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = [\underline{a} \otimes \underline{b}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = [\underline{a}]_{3\times1} \left([\underline{b}]_{3\times1}\right)^T. \tag{1.10}$$

Notice that the tensor product implies that the second vector is transposed. This is in contrast with the dot product where the first vector is transposed. The above definition implies that the representation of this second order tensor $\underline{\underline{\mathrm{C}}}$ is a matrix whose individual components are given by

$$C_{ij} = a_i \, b_j.$$

1.3 Second order tensors and their representation

The tensor product could also be written as follows:

$$\underline{\underline{C}} = \underline{a} \otimes \underline{b} \;\; = \;\; \sum_i a_i \, \underline{e}_i \otimes \sum_j b_j \, \underline{e}_j = \sum_i \sum_j a_i b_j \, \underline{e}_i \otimes \underline{e}_j.$$

Notice that we can move scalars anywhere in an expression, e.g., we moved b_j from right to left of the tensor product symbol and further to the left of vector $\underline{e}_i$. We cannot do the same with vectors, e.g., we cannot change the order of appearance of the two vectors. Upon contrasting the above form with the expansion of a vector in equation (1.4), we make a note that, just like a general vector is expressed as a linear combination of three basis vectors, a second order tensor can be expressed as a linear combination of nine basis tensors. Each of the basis tensors ($\underline{e}_i \otimes \underline{e}_j$) here are themselves tensors. A general second order tensor can be written as follows:

$$\underline{\underline{C}} = \sum_i \sum_j C_{ij}\ \underline{e}_i \otimes \underline{e}_j. \tag{1.11}$$

The nine coefficients in C_{ij} are, in general, independent of each other. The coefficient C_{ij} can be thought of as the component of the tensor $\underline{\underline{C}}$ along the basis tensor $\underline{e}_i \otimes \underline{e}_j$. Using (1.10), we can also write the following for one of the basis tensors:

$$\left[\underline{e}_1 \otimes \underline{e}_2\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.12}$$

Thus, just like basis vectors $\underline{e}_i$ have column form, each of the basis tensors has a matrix form. Using (1.11) and (1.12), it is easy to see that the coefficient C_{ij} in (1.11) also forms the i^{th} row and j^{th} column of the matrix representation of $\underline{\underline{C}}$ in ($\underline{e}_1, \underline{e}_2, \underline{e}_3$) coordinate system. Just like vectors, the matrix form of a tensor changes from one coordinate system to other but the tensor itself does not change, e.g.,

$$\underline{\underline{C}} = \sum_i \sum_j C_{ij}\ \underline{e}_i \otimes \underline{e}_j = \sum_i \sum_j \hat{C}_{ij}\ \hat{\underline{e}}_i \otimes \hat{\underline{e}}_j.$$

$$\downarrow \ \text{(matrix form)} \ \downarrow$$

$$[C_{ij}] \qquad\qquad [\hat{C}_{ij}]$$

Thus, we get different matrices in different coordinate system but the tensor itself is still the same. As a final remark, tensors can be of any order which are all independent of the coordinate system but their representations change from one coordinate system to the other. For example, all scalars are zeroth order tensors whereas all vectors are first order tensors. We can also have third and fourth (even higher) order tensors. A common question often comes to our mind. There are several examples of vectors in real life (velocity, force etc.). Do tensors exist in reality or are they abstract quantities? We will have a clearer answer to this question as we progress through the book. We also have the concept of transpose of a tensor. For a second order tensor $\underline{a} \otimes \underline{b}$, its *transpose* is defined to be $\underline{b} \otimes \underline{a}$. For a general second order tensor of the form

$$\underline{\underline{A}} = \sum_i \sum_j A_{ij}\ \underline{e}_i \otimes \underline{e}_j, \tag{1.13}$$

we can write its transpose to be

$$\underline{\underline{A}}^T = \sum_i \sum_j A_{ij}\ \underline{e}_j \otimes \underline{e}_i. \tag{1.14}$$

Basically, the order of two vectors in tensor product operation is flipped. Upon further renaming $i \to j$ and $j \to i$, we get

$$\underline{\underline{A}}^T = \sum_j \sum_i A_{ji}\ \underline{e}_i \otimes \underline{e}_j. \tag{1.15}$$

From definitions (1.13) and (1.15), one can then realize that the matrix form of transpose of $\underline{\underline{A}}$ is equal to the transpose of the matrix form of $\underline{\underline{A}}$, i.e.,

$$[\underline{\underline{A}}^T] = [\underline{\underline{A}}]^T.$$

1.4 Multiplying a second order tensor with a vector: dot product and tensor product

Just like there are several ways in which two vectors can be multiplied (scalar product, vector product, tensor product), the same can be said about multiplication of a second order tensor with a vector.

We start with the dot product of a tensor with a vector. It is also called *single contraction* and is defined as follows:

$$\underline{a} = \underline{\underline{C}} \cdot \underline{b} = \left(\sum_i \sum_j C_{ij} \; \underline{e}_i \otimes \underline{e}_j \right) \cdot \left(\sum_k b_k \; \underline{e}_k \right) = \sum_i \sum_j \sum_k C_{ij} \; b_k \; \left(\underline{e}_i \otimes \underline{e}_j \right) \; \cdot \underline{e}_k.$$

Often, the dot symbol is suppressed when we mean this single contraction. To proceed further, the second vector from the 2nd order tensor is dotted with the first order tensor, i.e.,

$$\underline{\underline{C}} \cdot \underline{b} = \sum_i \sum_j \sum_k C_{ij} \; b_k \; \underline{e}_i \; (\underline{e}_j \cdot \underline{e}_k) = \sum_i \sum_j \sum_k C_{ij} \; b_k \; \underline{e}_i \; \delta_{jk}. \tag{1.16}$$

Here δ_{jk} is called the Kronecker delta function and is defined as follows:

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{1.17}$$

Now consider the summation over k in (1.16). Due to the Kronecker delta function present there, only the term having $k = j$ will contribute to the summation and others will be zero. Thus, we can get rid of the summation over k and replace k by j at all places, i.e.,

$$\underline{a} = \underline{\underline{C}} \cdot \underline{b} = \sum_i \left(\sum_j C_{ij} b_j \right) \underline{e}_i.$$

As any general vector can be written as $\underline{a} = \sum_i a_i \; \underline{e}_i$, the term in the parentheses above turns out to be a_i. Thus, when we dot product/single contract a second order tensor with a vector, we get a vector whose components are given by

$$a_i = \sum_j C_{ij} b_j = \begin{bmatrix} C_{i1} & C_{i2} & C_{i3} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \left[\underline{\underline{C}}\right] \left[\underline{b}\right]. \tag{1.18}$$

i^{th} *row of* $\left[\underline{\underline{C}}\right]$

Thus, we simply multiply the matrix form of $\underline{\underline{C}}$ with the column form of $\underline{b}$ in the usual way to get the column form of the resulting vector $\underline{a}$. At this point, let us recall the cross product definition in (1.7) where we had written it as a skew symmetric matrix times a vector. On further noting the multiplication we just saw in (1.18), we immediately conclude that the cross product of two vectors can also be thought of as a second order tensor dotted with the second vector where the second order tensor corresponds to the first vector, i.e.,

$$\underset{\sim}{c} = \underset{\sim}{a} \times \underset{\sim}{b} = \underset{\approx}{a} \cdot \underset{\sim}{b}. \tag{1.19}$$

We also have tensor product of a second order tensor with a vector: the outcome is a third order tensor. It is defined as follows:

$$\underset{\equiv}{a} = \underset{\approx}{C} \otimes \underset{\sim}{b} = \left(\sum_i \sum_j C_{ij}\ \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\right) \otimes \left(\sum_k b_k\ \underset{\sim}{e}_k\right) = \sum_i \sum_j \sum_k C_{ij}\ b_k\ \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k. \tag{1.20}$$

Extracting the coefficients in the matrix representation of a second order tensor

To get the coefficient, say C_{kl}, of the matrix form of a tensor $\underset{\approx}{C}$ in $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ coordinate system, we'll verify that

$$C_{kl} = (\underset{\approx}{C} \cdot \underset{\sim}{e}_l) \cdot \underset{\sim}{e}_k. \tag{1.21}$$

Let us write the above equation in $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ coordinate system for $k = 1, l = 2$:

$$(\underset{\approx}{C} \cdot \underset{\sim}{e}_2) \cdot \underset{\sim}{e}_1 = \left(\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_{12} \\ C_{22} \\ C_{32} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = C_{12}.$$

This verifies equation (1.21). Stated differently, by using equation (1.21), we are able to extract the component of an arbitrary second order tensor $\underset{\approx}{C}$ relative to the basis tensor $\underset{\sim}{e}_k \otimes \underset{\sim}{e}_l$.

1.5 Multiplying two second order tensors: single-contraction, double-contraction and tensor product

There are again various ways in which two second order tensors can be operated together. We begin with single contraction. The single contraction of two second order tensors yields another second order tensor. This single contraction is defined as follows:

$$\begin{aligned}
\underset{\approx}{C} &= \underset{\approx}{a} \cdot \underset{\approx}{b} = \left(\sum_i \sum_j a_{ij}\ \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\right) \cdot \left(\sum_k \sum_l b_{kl}\ \underset{\sim}{e}_k \otimes \underset{\sim}{e}_l\right) \\
&= \sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl}\ (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) \cdot (\underset{\sim}{e}_k \otimes \underset{\sim}{e}_l) \\
&\qquad \text{(the inner ones } \underset{\sim}{e}_j \text{ and } \underset{\sim}{e}_k \text{ are dotted by definition)} \\
\Rightarrow \underset{\approx}{C} &= \sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl}\ (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_l)\ \delta_{jk}.
\end{aligned} \tag{1.22}$$

Using Kronecker delta property, we then remove the summation over j and replace j with k, i.e.,[2]

$$\underset{\approx}{C} = \sum_i \sum_k \sum_l a_{ik} b_{kl}\ \underset{\sim}{e}_i \otimes \underset{\sim}{e}_l = \sum_i \sum_l \left(\sum_k a_{ik} b_{kl}\right) \underset{\sim}{e}_i \otimes \underset{\sim}{e}_l.$$

[2] Since we have summation over both j and k in (1.22), we could have alternately removed the summation over k instead of j and replaced k with j everywhere also.

This is the expression for $\underline{\underline{C}}$ and a general second order tensor $\underline{\underline{C}}$ can be written as $\sum_i \sum_l C_{il}\ \underline{e}_i \otimes \underline{e}_l$ (using equation (1.11)). Noting this in the above expression, we see that the expression within the bracket is C_{il}, i.e.,

$$C_{il} \quad = \quad \sum_k a_{ik} b_{kl} = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} \end{bmatrix} \begin{bmatrix} b_{1l} \\ b_{2l} \\ b_{3l} \end{bmatrix}.$$

$$i^{th}\ row\ of\ \left[\underline{\underline{a}}\right],\ l^{th}\ column\ of\ \left[\underline{\underline{b}}\right]$$

Writing this for all components together, we get

$$\left[\underline{\underline{C}}\right] = \left[\underline{\underline{a}}\right] \left[\underline{\underline{b}}\right]. \tag{1.23}$$

Thus, we see that when two second order tensors are single-contracted, their matrix forms multiply in the usual way: the matrix form of the resultant tensor is simply the multiplication of the matrix forms of individual tensors.

The double contraction of two second order tensors yields a scalar and is defined as follows:

$$\begin{aligned} c = \underline{\underline{a}} : \underline{\underline{b}} &= \left(\sum_i \sum_j a_{ij}\ \underline{e}_i \otimes \underline{e}_j\right) : \left(\sum_k \sum_l b_{kl}\ \underline{e}_k \otimes \underline{e}_l\right) \\ &= \sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl}\ (\underline{e}_i \otimes \underline{e}_j) : (\underline{e}_k \otimes \underline{e}_l). \end{aligned}$$

By definition, the corresponding vectors on two sides of "double contraction" are dotted, i.e., $\underline{e}_i$ with $\underline{e}_k$ and $\underline{e}_j$ with $\underline{e}_l$. We thus obtain

$$c = \sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl} \delta_{ik} \delta_{jl} = \sum_i \sum_j a_{ij} b_{ij}. \tag{1.24}$$

Basically, the corresponding components of two tensors are multiplied and summed - this is very much like we do in dot product of two vectors.

We then talk of tensor product of two second order tensors. It yields a fourth order tensor as follows:

$$\begin{aligned} \underline{\underline{\underline{\underline{c}}}} = \underline{\underline{a}} \otimes \underline{\underline{b}} \quad &= \quad \left(\sum_i \sum_j a_{ij}\ \underline{e}_i \otimes \underline{e}_j\right) \otimes \left(\sum_k \sum_l b_{kl}\ \underline{e}_k \otimes \underline{e}_l\right) \\ &= \quad \sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl}\ \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l. \end{aligned} \tag{1.25}$$

1.6 Calculus of tensors: divergence and gradient

The divergence of a tensor is defined as follows:

$$\underline{\nabla} \cdot (\circ) = \sum_i \frac{\partial}{\partial x_i} (\circ) \cdot \underline{e}_i.$$

For example, for a first order tensor:

$$\underline{\nabla} \cdot \underline{v} = \sum_i \frac{\partial}{\partial x_i} (\underline{v}) \cdot \underline{e}_i = \sum_i \frac{\partial}{\partial x_i} \left(\sum_j v_j \underline{e}_j \right) \cdot \underline{e}_i = \sum_i \sum_j \frac{\partial v_j}{\partial x_i} \delta_{ij} = \sum_i \frac{\partial v_i}{\partial x_i}.$$

The gradient of a tensor is defined as follows:

$$\underline{\nabla} (\circ) = \sum_i \frac{\partial}{\partial x_i} (\circ) \otimes \underline{e}_i.$$

Notice its difference with the formula for the divergence of a tensor. The gradient of a vector $\underline{f}$, e.g., is

$$\underline{\nabla}\, \underline{f} = \sum_j \frac{\partial \underline{f}}{\partial X_j} \otimes \underline{e}_j = \sum_i \sum_j \frac{\partial f_i}{\partial X_j} \underline{e}_i \otimes \underline{e}_j. \tag{1.26}$$

It is a second order tensor whose matrix form is simply formed by the coefficient $\frac{\partial f_i}{\partial X_j}$. Similarly, the gradient of a scalar quantity is defined as follows:

$$\underline{\nabla}\, f = \sum_i \frac{\partial f}{\partial X_i} \underline{e}_i. \tag{1.27}$$

If we represent this vector in the coordinate system $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$, we get

$$\left[\underline{\nabla}\, f\right]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} = \begin{bmatrix} \frac{\partial f}{\partial X_1} & \frac{\partial f}{\partial X_2} & \frac{\partial f}{\partial X_3} \end{bmatrix}^T. \tag{1.28}$$

This is the column from of gradient that we have seen in first year math course. Notice that divergence reduces the order of a tensor by 1 whereas gradient increases the order of a tensor by 1.

1.7 Rotation tensor

As the name suggests, rotation tensors are related to physical rotation of objects. They can be used to rotate vectors as well as tensors. It should have the property such that after rotation, vectors and tensors do not change their magnitude but only direction. Let us consider two sets of orthonormal triads (see Figure 1.4: each set contains three vectors which are perpendicular to each other and also of unit magnitude). One can always transform a set of triad into another through a unique rotation or what we call a rotation tensor. Mathematically, we write

$$\hat{\underline{e}}_i = \underline{\underline{R}}\, \underline{e}_i \;\forall\, i = 1, 2, 3. \tag{1.29}$$

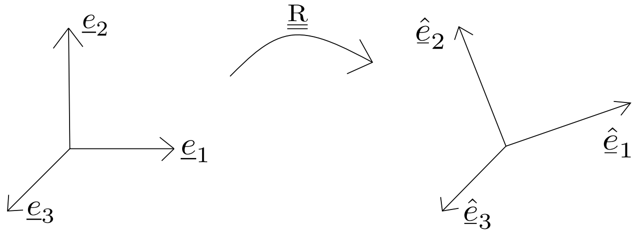


Figure 1.4: Two sets of orthonormal triads related through a rotation tensor

The matrix form of this rotation tensor turns out to be an orthonormal ('ortho' means perpendicular and 'normal' means normalized) matrix. The tensor itself is called an orthonormal tensor. Orthonormal tensors and their matrix forms have the following properties:

$$(a)\ \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}} \text{ (an identity tensor)}, \quad (b)\ det(\underline{\underline{R}}) = 1. \tag{1.30}$$

$$\Rightarrow \quad [\underline{\underline{R}}]\ [\underline{\underline{R}}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(matrix form of (1.30a))}$$

$$\Rightarrow \quad \sum_k R_{ik}R_{jk} = \delta_{ij}\ \forall i,j = 1,2,3 \quad \text{(using indical notation)} \tag{1.31}$$

This means that rows of $[\underline{\underline{R}}]$ are perpendicular to each other and are themselves normalized to be of unit magnitude, i.e., the rows are orthonormal with respect to each other. One can similarly prove that its columns are also orthonormal with respect to each other.

Let us consider an specific example and see what does a rotation matrix look like. In Figure 1.5,

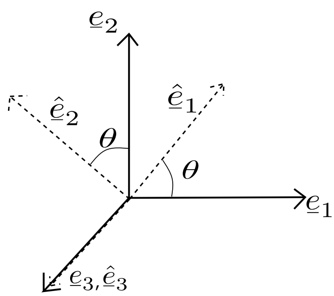


Figure 1.5: A basis triad rotated by an angle θ about $\underline{e}_3$

we have $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system and it is rotated by $\underline{\underline{R}}$ to get the new coordinate system $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$. The rotation is such that $\hat{\underline{e}}_3$ is same as $\underline{e}_3$ and the other two basis vectors are rotated by θ. Let us determine the matrix form $[\underline{\underline{R}}]$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. We know from equation (1.21) that

$$R_{ij} = (\underline{\underline{R}}\ \underline{e}_j)\cdot\underline{e}_i = \hat{\underline{e}}_j\cdot\underline{e}_i \quad \text{(using (1.29))} \tag{1.32}$$

So, the j^{th} column of the rotation matrix is the column form of vector $\hat{\underline{e}}_j$ expressed in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. For example, consider j=2:

$$R_{i2} = \hat{\underline{e}}_2 \cdot \underline{e}_i.$$

We can thus find these coefficients in terms of θ using dot product definition, e.g.,

$$\hat{\underline{e}}_2 \cdot \underline{e}_1 = -\text{sin}\theta, \quad \hat{\underline{e}}_2 \cdot \underline{e}_2 = \text{cos}\theta, \quad \hat{\underline{e}}_2 \cdot \underline{e}_3 = 0.$$

Working out all the components in this way, we get

$$\left[\underline{\underline{R}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \cos\theta & -sin\theta & 0 \\ sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the third column is $[0\ 0\ 1]^T$ as $\hat{\underline{e}}_3$ is same as $\underline{e}_3$. We emphasize that the above matrix form is the representation of rotation tensor $\underline{\underline{R}}$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. This matrix form will be different if we choose another coordinate system to represent the tensor.

As a side note, any rotation can be uniquely characterized by its axis of rotation (unit vector $\underline{a}$) and the angle of rotation θ. One can then use the following Rodrigue's formula to obtain the rotation tensor in terms of these two information:

$$\underline{\underline{R}}(\underline{a}, \theta) = \underline{\underline{I}} + \sin(\theta)\ \underline{\underline{a}} + (1 - \cos(\theta))\ \underline{\underline{a}}^2. \tag{1.33}$$

Here $\underline{\underline{a}}$ is a skew symmetric tensor whose axial vector is $\underline{a}$.

1.8 Solved examples

Q1. Show that $\underline{a} \cdot (\underline{\underline{A}}\ \underline{b}) = (\underline{\underline{A}}^T \underline{a}) \cdot \underline{b}$

Method 1 (using indical notation):

$$\begin{aligned}
\underline{a} \cdot (\underline{\underline{A}}\ \underline{b}) &= \sum_i a_i \underline{e}_i \cdot \left[\left(\sum_j \sum_k A_{jk}\ \underline{e}_j \otimes \underline{e}_k \right) \cdot \sum_l b_l \underline{e}_l \right] \\
&= \sum_i a_i \underline{e}_i \cdot \sum_j \sum_k \sum_l A_{jk} b_l \delta_{kl}\ \underline{e}_j \\
&= \sum_i \sum_j \sum_k \sum_l a_i A_{jk} b_l \delta_{kl} \delta_{ij} = \sum_j \sum_l a_j A_{jl} b_l
\end{aligned}$$

$$(\underline{\underline{A}}^T \underline{a}) \cdot \underline{b} = \left[\left(\sum_j \sum_k A_{kj}\ \underline{e}_j \otimes \underline{e}_k \right) \cdot \sum_i a_i \underline{e}_i \right] \cdot \sum_l b_l \underline{e}_l$$

$$= \sum_i \sum_j \sum_k a_i A_{kj} \delta_{ki} \, \underline{e}_j \cdot \sum_l b_l \underline{e}_l$$
$$= \sum_i \sum_j \sum_k \sum_l a_i A_{kj} b_l \delta_{ki} \delta_{lj}$$
$$= \sum_i \sum_j a_i A_{ij} b_j = \sum_j \sum_l a_j A_{jl} b_j \text{ (due to renaming of dummy indices)}$$

Method 2 (using their matrix forms):

$$\begin{aligned} \underline{a} \cdot (\underline{\underline{A}} \, \underline{b}) &= [\underline{a}]^T \, [\underline{\underline{A}}] \, \ [\underline{b}] \\ &= \left([\underline{a}]^T \, [\underline{\underline{A}}] \, \ [\underline{b}] \right)^T \quad (\because \text{ transpose of a scalar remains the same}) \\ &= [\underline{b}]^T \, [\underline{\underline{A}}]^T \, [\underline{a}] \\ &= [\underline{b}]^T \left([\underline{\underline{A}}]^T \, [\underline{a}] \right) \\ &= [\underline{b}] \cdot \left[\underline{\underline{A}}^T \underline{a} \right] = \underline{b} \cdot \left(\underline{\underline{A}}^T \underline{a} \right). \end{aligned}$$

Q2. There is a tensor $\underline{\underline{A}}$ such that $\underline{\underline{A}} \cdot \underline{e}_1 = \underline{a}$, $\underline{\underline{A}} \cdot \underline{e}_2 = \underline{b}$, $\underline{\underline{A}} \cdot \underline{e}_3 = \underline{c}$. What will be the matrix form of $\underline{\underline{A}}$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system?

Solution: As per information provided: $\underline{\underline{A}} \cdot \underline{e}_1 = \underline{a}$

$$\text{or} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{(expressed in the sought coordinate system)}$$
$$\therefore \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{(1st column of } [\underline{\underline{A}}] = [\underline{a}])$$

One can similarly show that

$$\begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ (2nd column of } [\underline{\underline{A}}] = [\underline{b}]) \text{ and } \begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \text{ (3rd column of } [\underline{\underline{A}}] = [\underline{c}]).$$

To see how the tensor form of $\underline{\underline{A}}$ looks like, one can derive as follows:

$$\begin{aligned} [\underline{\underline{A}}] &= \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & 0 & 0 \\ A_{31} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_{12} & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \\ &= [\underline{a}] \, [\underline{e}_1]^T + [\underline{b}] \, [\underline{e}_2]^T + [\underline{c}] \, [\underline{e}_3]^T \\ \text{or } \underline{\underline{A}} &= \underline{a} \otimes \underline{e}_1 + \underline{b} \otimes \underline{e}_2 + \underline{c} \otimes \underline{e}_3. \end{aligned}$$

Q3. Show that
(a) $(\underline{a} \otimes \underline{b})\, \underline{\underline{C}} = \underline{a} \otimes \left(\underline{\underline{C}}^T \underline{b}\right)$, (b) $\underline{\underline{C}}\, (\underline{a} \otimes \underline{b}) = \left(\underline{\underline{C}}\ \underline{a}\right) \otimes \underline{b}$

(a) Let's start with the matrix form of tensor expression $(\underline{a} \otimes \underline{b})\, \underline{\underline{C}}$

$$\left([\underline{a}]\ [\underline{b}]^T\right) [\underline{\underline{C}}] = [\underline{a}] \left([\underline{b}]^T\ [\underline{\underline{C}}]\right) \quad \text{(using associative rule of matrix multiplication)}$$
$$= [\underline{a}] \left([\underline{\underline{C}}]^T\ [\underline{b}]\right)^T = \underline{a} \otimes \left(\underline{\underline{C}}^T \underline{b}\right)$$

(b) Starting with the matrix form of tensor expression $\underline{\underline{C}}\, (\underline{a} \otimes \underline{b})$

$$\left[\underline{\underline{C}}\, (\underline{a} \otimes \underline{b})\right] = [\underline{\underline{C}}] \left([\underline{a}]\, [\underline{b}]^T\right) = \left([\underline{\underline{C}}]\ [\underline{a}]\right) [\underline{b}]^T = \left(\underline{\underline{C}}\ \underline{a}\right) \otimes \underline{b}.$$

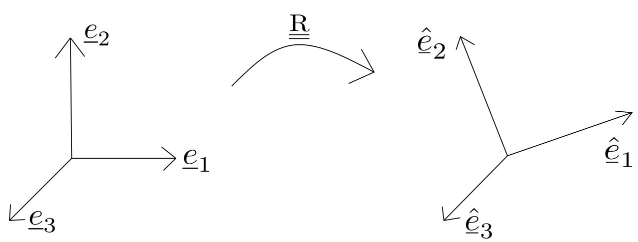
Q4. Given an anti-symmetric tensor $\underline{\underline{A}}$, prove that $\left(\underline{\underline{A}}\ \underline{x}\right) \cdot \underline{x} = 0\ \ \forall\ \underline{x}$

Solution:

$$\begin{aligned}\left(\underline{\underline{A}}\ \underline{x}\right) \cdot \underline{x} &= \underline{x} \cdot \left(\underline{\underline{A}}^T \underline{x}\right) \\ &= -\underline{x} \cdot \left(\underline{\underline{A}}\ \underline{x}\right) \\ &= -\left(\underline{\underline{A}}\ \underline{x}\right) \cdot \underline{x} \text{ (from commutative rule of dot-product of two vectors)} \\ &= 0 \text{ (since only zero can be negative of itself!)}\end{aligned}$$

Disclaimer: The commutative rule ($\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$) applies only for 1st-order tensors and does not extend to higher order tensors. For example, $\underline{\underline{A}} \cdot \underline{b} \neq \underline{b} \cdot \underline{\underline{A}}$ or $\underline{\underline{A}} \cdot \underline{\underline{C}} \neq \underline{\underline{C}} \cdot \underline{\underline{A}}$.

Q5. We have learnt that a unique rotation tensor $\underline{\underline{R}}$ can be associated with transforming a set of orthonormal triad into another say $(\underline{e}_1, \underline{e}_2, \underline{e}_3) \rightarrow (\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$. Find the matrix form of this rotation tensor $\underline{\underline{R}}$ in $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$ coordinate system.



Solution: Recall that a tensor $\underline{\underline{R}}$ remains invariant in different coordinate systems. However, its representation or its matrix form is different in different coordinate systems. Also, rotation tensors are orthonormal and satisfy the following property: $\underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$. Since $\underline{\underline{R}}$ in the question maps $(\underline{e}_1, \underline{e}_2, \underline{e}_3) \rightarrow (\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$, we can write

$$\hat{\underline{e}}_i = \underline{\underline{R}}\ \underline{e}_i \Rightarrow \underline{\underline{R}}^{-1} \hat{\underline{e}}_i = \underline{e}_i \text{ or } \underline{\underline{R}}^T \hat{\underline{e}}_i = \underline{e}_i. \tag{1.34}$$

One can also write

$$\underline{\underline{R}} = \sum_i \sum_j \hat{R}_{ij}\ \underline{\hat{e}}_i \otimes \underline{\hat{e}}_j$$

where the components $\hat{R}_{ij}$ can be obtained as follows:

$$\hat{R}_{ij} = \left(\underline{\underline{R}}\ \underline{\hat{e}}_j\right) \cdot \underline{\hat{e}}_i = \underline{\hat{e}}_j \cdot \underline{\underline{R}}^T \underline{\hat{e}}_i = \underline{\hat{e}}_j \cdot \underline{e}_i \quad \text{(using (1.34))}$$

We had derived the same expression for R_{ij} earlier while discussing rotation tensors. This result is non-intuitive as we do not expect the representation of a tensor in two different coordinate systems to be the same in general. However, this case happens to be a special one since the given tensor also transforms the first coordinate system into other.

1.9 Objective questions to recall concepts

(1) We can say the following about a tensor:
  (a) It is independent of the coordinate system
  (b) It changes from one coordinate system to other
  (c) Its representation changes from one coordinate system to another
  (d) None of the above

(2) Which of the following statements is/are correct?
  (a) Scalars are zeroth order tensors
  (b) Tensors are independent of the coordinate system
  (c) Vectors are more general than tensors
  (d) One can add tensors of different orders

(3) What is the product of a scalar with another vector?
  (a) It's a vector
  (b) It's a second order tensor
  (c) It's a scalar
  (d) None of these

(4) What is the product of a vector with another vector?
  (a) It's a vector
  (b) It depends on whether we are doing dot product, cross-product or tensor product
  (c) Dot product gives a scalar quantity
  (d) None of these

(5) Which of the following statements is/are correct concerning the cross product of two vectors?
  (a) Cross product of two vectors gives us a second order tensor.
  (b) Cross product of two vectors is the same as the product of a skew-symmetric tensor formed by the first vector times the second vector.
  (c) Cross product of two vectors is dependent on the coordinate system.
  (d) None of these

(6) A second order tensor, in general, has
  (a) three independent coefficients
  (b) six independent coefficients

(c) nine independent coefficients
(d) none of these

(7) What is the order of the tensor obtained from the dyadic product of two vectors?
(a) Zeroth order
(b) First order
(c) Second order
(d) None of these

(8) Which of the following is/are incorrect concerning vectors?
(a) The representation of a vector is different in different coordinate systems.
(b) A vector is a second order tensor.
(c) A vector is a first order tensor.
(d) A vector remains the same irrespective of change in the coordinate system.

(9) We can say the following about a rotation tensor.
(a) It uniquely relates a set of three orthonormal vectors with another set of three orthonormal vectors
(b) Any two columns of its matrix form are orthogonal to each other
(c) The determinant of its matrix form equals -1
(d) Its inverse equals its transpose

(10) Which of the following is/are the property of a rotation tensor $\underline{\underline{R}}$?
(a) $\underline{\underline{R}}\underline{\underline{R}}^T = \underline{\underline{I}}$
(b) $\det(\underline{\underline{R}}) = 1$
(c) $\det(\underline{\underline{R}}) = -1$
(d) $\underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{I}}$

(11) Which of the following is true for the tensor equation: $\underline{a} = \underline{\underline{R}}\, \underline{b}$?
(a) It can be expressed/expanded only in $\underline{e}_1$-$\underline{e}_2$-$\underline{e}_3$ coordinate system
(b) It can be expressed/expanded in any coordinate system
(c) It can be expressed/expanded only in that coordinate system in which we are working
(d) None of these

1.10 Questions to practise

(1) If $\underline{a}$ and $\underline{b}$ are two vectors, show that their dot product $\underline{a} \cdot \underline{b}$ remain the same in different coordinate systems (say $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ and $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$).
(2) Prove that $\left[(\underline{u} \cdot \underline{a}) \left(\underline{\underline{A}}^T \underline{b}\right)\right] \cdot \underline{n} = \underline{a} \cdot \left[\left(\underline{u} \otimes \underline{\underline{A}}\underline{n}\right)\right] \underline{b}$.
(3) If $\underline{\underline{A}}$ is a symmetric tensor and $\underline{\underline{B}}$ is an anti-symmetric tensor, then show that

$$\underline{\underline{A}} : \underline{\underline{B}} = 0.$$

Chapter 2

Traction Vector, Stress Tensor and Stress Matrix

In this chapter, we will learn the concepts of traction vector, stress tensor and its matrix representation. We will also learn how to transform stress matrix corresponding to one coordinate system into another.

2.1 Definition of traction vector

Consider an arbitrary body which is clamped at one part of the boundary and some external load is applied elsewhere on its boundary (see Figure 2.1). The clamped boundary of the body does not move but the remaining body deforms due to the applied load. In the deformed state, we say that the body is

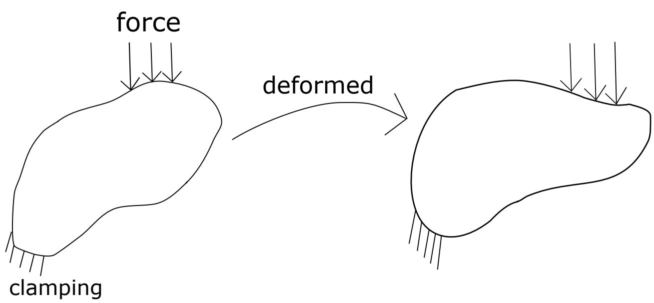


Figure 2.1: A body deforming under the action of an applied load

under some stress. Let us imagine cutting the deformed body into two parts A and B through an internal section as shown by the dashed lines in Figure 2.2. Let us analyze part A. The part B exerts some force on part A through the internal section - this force is distributed over the internal section as shown in

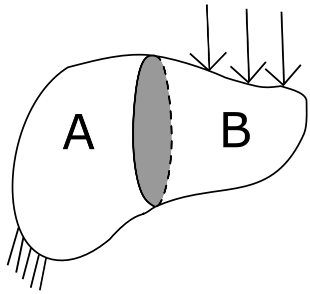


Figure 2.2: An internal section (shown shaded) in the deformed body dividing it into two parts

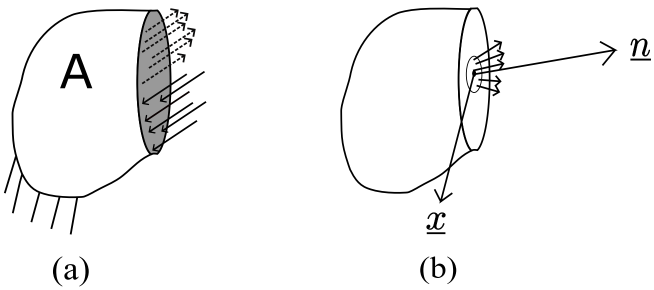


Figure 2.3: (a) Part A of the body when considered alone is acted upon by forces applied by Part B (b) a small circular area around a point $\underline{x}$ on the internal section having outward normal $\underline{n}$

Figure 2.3a. We define traction to be the intensity of the force (force per unit area) with which Part B is pulling or pushing part A. At different points in the section, this force intensity would be different in both magnitude and direction. For example, the upper portion of the section is being pulled while the bottom portion is being pushed. To find traction at some point $\underline{x}$ on the section (see Figure 2.3b), we draw a circle around this point in the plane of the section. Let the area enclosed by the circle be ΔA and the net force acting on this circular area be $\Delta\overrightarrow{F}$. We then shrink the area of the circle such that the circle always contains $\underline{x}$. As we shrink the area, the total force acting on it decreases but the ratio of this force and the area of the enclosing circle attains a limiting value which is called traction. We represent it by $\underline{t}(\underline{x};\underline{n})$ where $\underline{n}$ denotes the normal to the section but pointing outward/away from part A on which it is acting. Mathematically, we write

$$\underline{t}(\underline{x};\underline{n}) = \lim_{\Delta A \to 0} \frac{\Delta\overrightarrow{F}}{\Delta A}. \tag{2.1}$$

Although it has the unit of pressure, it is more general than pressure: pressure always acts in the direction opposite to the outward plane normal whereas traction can act in an arbitrary direction.

Dependence of traction on plane normal

Think of an arbitrary body under the influence of external load and consider three points inside it (see Figure 2.4). In general, traction will be different at these three points. At a given point also, say $\underline{x}$,

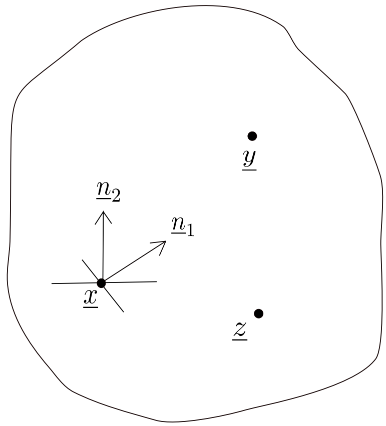


Figure 2.4: Three arbitrary points ($\underline{x}$, $\underline{y}$, $\underline{z}$) shown in a body along with two different internal sections/planes at point $\underline{x}$ having normals $\underline{n}_1$ and $\underline{n}_2$

several planes/internal sections can be imagined and traction on each of these planes will be different. In Figure 2.4 for example, traction on the plane having normal $\underline{n}_1$ will be different from traction on plane having normal $\underline{n}_2$. To see this more clearly, let us consider a rectangular beam which is clamped at one end (see Figure 2.5). A force P is being applied to its other end. The cross-sectional area of the beam is A. From force balance of the entire beam, we can show that the clamped support at the left end exerts a force $-P\underline{e}_1$ to the leftmost end of the beam. Let us now consider an internal section of the

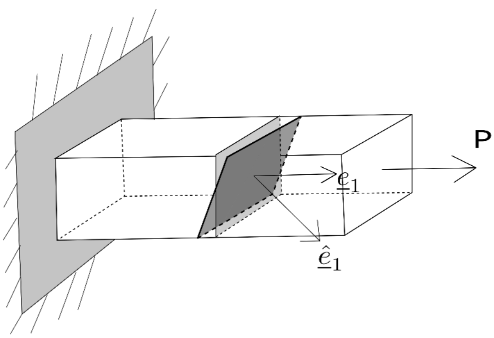


Figure 2.5: A beam acted upon by a force P: two internal sections shown

beam whose normal vector (denoted by $\underset{\sim}{e}_1$) is along the axis of the beam. This section divides the beam into two parts. From the force balance of left portion of the beam, one can show that the total internal force on this section also equals $P\underset{\sim}{e}_1$. Hence, the traction on this plane (denoted by $\underset{\sim}{t}^1$) is given by

$$\underset{\sim}{t}^1 = \frac{P\underset{\sim}{e}_1}{A} = \frac{P}{A}\underset{\sim}{e}_1.$$

Notice that traction is acting only along the plane normal and hence we call it the normal traction.[1] Now, let us take an inclined internal section (as shown in Figure 2.5) but at the same axial location as the vertical plane. The unit normal to this new section ($\hat{\underset{\sim}{e}}_1$) makes an angle θ with the beam's axis. So, the area of this section $\hat{A}$ is given by

$$\hat{A} = \frac{A}{\cos\theta}. \tag{2.2}$$

From force balance of the new left portion of the beam, one can show that the total force on the inclined section is still P$\underset{\sim}{e}_1$. So, the traction on this section (denoted by $\underset{\sim}{t}^{\hat{1}}$) will be

$$\underset{\sim}{t}^{\hat{1}} = \frac{P\underset{\sim}{e}_1}{A/\cos\theta} = \frac{P\cos\theta}{A}\underset{\sim}{e}_1.$$

Thus, we see that traction on two different planes at the same point are different. Also note that the traction vector now is not aligned with the plane normal. Its component along the plane normal is called the normal component of traction and equals $\frac{P}{A}\cos^2\theta$ in this case. On the other hand, the component of traction perpendicular to the plane normal is called the shear component of traction and equals $\frac{P}{A}\cos\theta\sin\theta$. Note that the shear component becomes maximum for the section for which $\theta = 45°$.

Importance of traction

By definition, it gives us the intensity of force with which one part of the body is pulling or pushing the other part of the body. If the value of this traction is lower than a threshold limit, the body will not fracture/fail. At a given point, as traction varies from one plane to the other, the probability of failure is higher on the plane on which traction has got larger value. Thus, traction can tell us at what point in the body and on what plane at that point would the body fail! For example, it has been experimentally observed during tensile loading of a steel bar that at a critical value of applied load, the steel bar develops a crack in a plane that is inclined at around 45° to the direction of loading. This is because on this plane, the magnitude of shear component of traction is highest as we saw earlier. We clarify at this point that failure due to shear component of traction reaching a critical value is just one mechanism of failure and different materials obey different mechanisms of failure.

2.2 Relating traction on different planes at a point

We will now prove that if we know traction on three mutually perpendicular planes at a point, we can find traction on any plane at that point. Let us consider a point $\underset{\sim}{x}$ in the body as shown in Figure 2.6 and imagine a small volume there in the shape of a tetrahedron with its vertex at $\underset{\sim}{x}$. The tetrahedron's three edges at this point are assumed to be perpendicular to each other and lying along the three coordinate axes. This tetrahedron has four faces having the following outward normals (by outward we mean

[1] For simplicity, we have assumed here that we have the same traction at every point on this section although that is not the usual case.

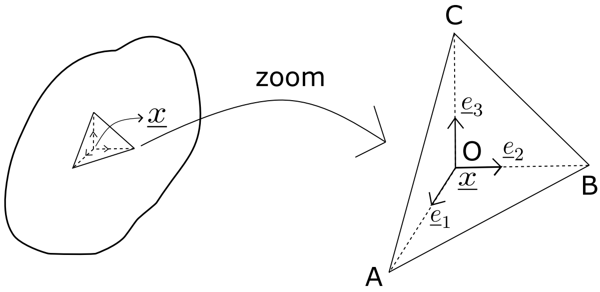


Figure 2.6: A small tetrahedron considered at a point of interest within the body

pointing out of the tetrahedron volume):

$\underline{Plane}$	:	$\underline{Outward\, normal}$
OAB	:	$-\underline{e}_3$
OBC	:	$-\underline{e}_1$
OAC	:	$-\underline{e}_2$
ABC	:	$\underline{n}$

Suppose, we know traction on three of these planes having outward normals $(-\underline{e}_1, -\underline{e}_2, -\underline{e}_3)$ and we want to find traction on the tilted plane ABC (having normal say $\underline{n}$). Let us apply Newton's 2^{nd} law of motion to the mass contained in the tetrahedron:

$$\sum \underline{F}^{ext} = \frac{d}{dt}(\overrightarrow{P}). \tag{2.3}$$

Here $\overrightarrow{P}$ denotes the linear momentum of the tetrahedron's mass. To visualize the external forces acting on the tetrahedron, we can imagine taking out the tetrahedron from the original body as shown on the right of Figure 2.6. The remaining part of the body exerts traction force on the four faces of the tetrahedron which are also categorized as surface force. However, the total external force consists of both surface and body forces:

$$\sum \underline{F}^{ext}$$

$$\swarrow \qquad\qquad \searrow$$

$$(surface\ \ force) \qquad\qquad (body\ \ force)$$

unit: [force/area]   unit: [force/volume]

The body force acts on every point of the body, e.g., the gravitational force is applied by the Earth on every point of the tetrahedron. Its unit is force/volume as it acts on the entire volume in a distributed manner and is given by

$$\text{Gravitational body force} = (\rho \underline{g} V)/V = \rho \underline{g}. \tag{2.4}$$

Here ρ denotes density, $\underline{g}$ is acceleration due to gravity and V is volume of the tetrahedron. To obtain the total surface force, we need to integrate traction forces over the area on which they are acting whereas to get the total body force, we need to integrate body force over the volume of the tetrahedron. Let the area of face OBC be A_1 and traction on it be $\underline{t}^{-1}$, area of face OAC be A_2 and traction on it be $\underline{t}^{-2}$, area of face OAB be A_3 and traction on it be $\underline{t}^{-3}$ and finally the area of face ABC be A_n and traction on it be $\underline{t}^{n}$. The total external force on the tetrahedron can be written as

$$\sum \underline{F}^{ext} = \underbrace{\underline{t}^{-1}_{avg}A_1 + \underline{t}^{-2}_{avg}A_2 + \underline{t}^{-3}_{avg}A_3 + \underline{t}^{n}_{avg}A_n}_{\text{contact force}} + \underbrace{\rho \underline{g} V}_{\text{non-contact force}} = \underbrace{\rho V}_{\text{tetrahedron mass}} \times \underline{a}_{CM}. \tag{2.5}$$

Here $\underline{a}_{CM}$ denotes the acceleration of center of mass of the tetrahedron. The subscript "avg" on four tractions is used to denote that they are the average values of traction on corresponding faces. Suppose h is the perpendicular distance of the plane ABC from the tetrahedron's vertex. The volume of the tetrahedron is then given by

$$V = \frac{A_n h}{3}. \tag{2.6}$$

We can also relate the areas A_1, A_2, A_3 in terms of A_n. If we project the area A_n along the direction $\underline{e}_1$, it will turn out to be the same as the area of OBC which is A_1. We can thus write

$$A_i = A_n(\underline{n} \cdot \underline{e}_i) \ \forall i = 1, 2, 3. \tag{2.7}$$

Upon plugging equations (2.6) and (2.7) into equation (2.5), we then get

$$A_n \left(\underline{t}^{-1}_{avg}(\underline{n} \cdot \underline{e}_1) + \underline{t}^{-2}_{avg}(\underline{n} \cdot \underline{e}_2) + \underline{t}^{-3}_{avg}(\underline{n} \cdot \underline{e}_3) + \underline{t}^{n}_{avg}\right) + \frac{\rho \underline{g} A_n h}{3} = \frac{\rho A_n h\, \underline{a}_{CM}}{3}$$

$$\Rightarrow \underline{t}^{n}_{avg} + \sum_{i=1}^{3} \underline{t}^{-i}_{avg}(\underline{n} \cdot \underline{e}_i) + \frac{\rho h(\underline{g} - \underline{a}_{CM})}{3} = 0. \tag{2.8}$$

Here we obtained the last equation upon dividing by the tilted area A_n. Note that our original plan was to obtain traction on an arbitrary plane $\underline{n}$ at a point in terms of traction on three other planes at the same point. However, for our tetrahedron example, three planes pass through the point O but the plane with normal $\underline{n}$ (the tilted plane) does not pass through the point O. So, we need to shrink the tetrahedron such that in the limiting case, all the four planes pass through O. We can achieve this by letting the perpendicular distance h go to zero. When we apply this limit ($\lim_{h\to 0}$) to equation (2.8), the terms proportional to h vanish. Furthermore, the average tractions become the corresponding traction values at the point $\underline{x}$ itself. Thus, in the limit of tetrahedron shrinking to point $\underline{x}$ where all the four planes exist now, the terms corresponding to body force and acceleration vanish and we get the following desired result:

$$\underline{t}^{n}(\underline{x}) = -\sum_{i=1}^{3} \underline{t}^{-i}(\underline{x})\ (\underline{n} \cdot \underline{e}_i). \tag{2.9}$$

Relating tractions on planes having opposite normals

By definition, $\underline{t}^{-i}$ is the traction on $-\underline{e}_i$ plane whereas $\underline{t}^{i}$ is the traction on $\underline{e}_i$ plane. The two tractions basically act on the same plane but with normals pointing in opposite direction. Let us understand it in the context of an internal section in the body from Figure 2.2. We redraw the two parts so created

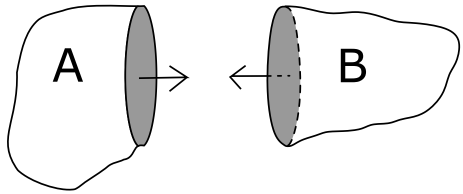


Figure 2.7: The common plane of two parts of a body having oppositely directed outward normals

in Figure 2.7. The original internal section now forms external surface of the two parts of the body and have outward normals pointing opposite to each other.[2] The tractions on these two planes (the two planes were coincident in the original body) will be equal and opposite due to Newton's third law since they form action-reaction pair. We can thus write $\underline{t}^{-i} = -\underline{t}^{i}$ substituting which in equation (2.9) yields

$$\boxed{\underline{t}^{n}(\underline{x}) = \sum_{i=1}^{3} \underline{t}^{i}(\underline{x})\,(\underline{n}\cdot\underline{e}_i)} \tag{2.10}$$

We emphasize that the body force and acceleration terms dropped out from the above formula - no approximation was made! Thus, the above formula holds even if the body force is present or the body is accelerating! Physically, the traction vector $\underline{t}^{n}$ has to be the same irrespective of what three planes are used to find it, i.e.,

$$\underline{t}^{n}(\underline{x}) = \sum_{i=1}^{3} \underline{t}^{i}(\underline{x})\,(\underline{n}\cdot\underline{e}_i) = \sum_{i=1}^{3} \underline{t}^{\hat{i}}(\underline{x})\,(\underline{n}\cdot\underline{\hat{e}}_i). \tag{2.11}$$

In the first case, the three planes used have normals along $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ whereas in the second case, the three planes have normals along $\underline{\hat{e}}_1$, $\underline{\hat{e}}_2$, $\underline{\hat{e}}_3$.

2.3 Stress Tensor

Let us now write equation (2.11) in a slightly different way:

$$\underline{t}^{n}(\underline{x}) = \sum_{i=1}^{3} \underline{t}^{i}(\underline{x})\,(\underline{n}\cdot\underline{e}_i) = \sum_{i=1}^{3} \underline{t}^{i}(\underline{x})\,(\underline{e}_i\cdot\underline{n}). \tag{2.12}$$

The ith term in the above summation can be written as follows in matrix form:

$$\underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}_{3\times 1}}_{\underline{t}^{i}} \left(\underbrace{[\qquad]_{1\times 3}}_{\underline{e}_i^{T}} \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}_{3\times 1}}_{\underline{n}} \right)$$

[2]Note that we always consider outward normals.

As matrix multiplication is associative, we can also write it alternatively as

$$\Bigg(\underbrace{\underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}_{3\times1}}_{\underline{t}^i} \underbrace{\begin{bmatrix} & \end{bmatrix}_{1\times3}}_{\underline{e}_i^T}}_{\text{tensor product}}\Bigg) \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}_{3\times1}}_{\underline{n}} \tag{2.13}$$

However, note that $[\underline{a}][\underline{b}]^T$ is the matrix representation for tensor product $(\underline{a} \otimes \underline{b})$. Thus, the alternate representation given by (2.13) can be written as follows in tensor form:

$$\underline{t}^n(\underline{x}) = \underbrace{\sum_{i=1}^{3} (\underline{t}^i(\underline{x}) \otimes \underline{e}_i)}_{\text{stress tensor}} \cdot \underline{n}. \tag{2.14}$$

This is just a different viewpoint but the interesting point here is that the orientation $\underline{n}$ has been separated. One could have also obtained it directly from equation (2.12) using the definition of dot product of a second order tensor with a vector. The tensor that is multiplied with $\underline{n}$ in the above expression is called the *stress tensor* $\underline{\underline{\sigma}}$ and one can write

$$\underline{\underline{\sigma}}(\underline{x}) = \sum_{i=1}^{3} \underline{t}^i(\underline{x}) \otimes \underline{e}_i \text{ such that } \underline{t}^n(\underline{x}) = \underline{\underline{\sigma}}(\underline{x})\, \underline{n}. \tag{2.15}$$

Note that the stress tensor depends on $\underline{x}$ alone. Essentially, to obtain the stress tensor at a point, choose three independent planes at that point, find tractions on those planes, do their tensor product with corresponding plane normals and sum! The resulting stress tensor will be independent of what specific set of three planes we choose!

2.4 Stress matrix as a representation of stress tensor

Let us try to represent equation (2.15) in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. As we now know that the stress tensor is a second order tensor, its representation is going to be a matrix, i.e.,

$$[\underline{\underline{\sigma}}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \sum_{i=1}^{3} [\underline{t}^i]\, [\underline{e}_i]^T .$$

Here we have to represent $\underline{t}^i$ and $\underline{e}_i$ both in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system, i.e.,

$$\begin{aligned} [\underline{\underline{\sigma}}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} &= [\underline{t}^1]\begin{bmatrix}1 & 0 & 0\end{bmatrix} + [\underline{t}^2]\begin{bmatrix}0 & 1 & 0\end{bmatrix} + [\underline{t}^3]\begin{bmatrix}0 & 0 & 1\end{bmatrix} \\ &= \begin{bmatrix} t_1^1 & 0 & 0 \\ t_2^1 & 0 & 0 \\ t_3^1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & t_1^2 & 0 \\ 0 & t_2^2 & 0 \\ 0 & t_3^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t_1^3 \\ 0 & 0 & t_2^3 \\ 0 & 0 & t_3^3 \end{bmatrix} = \begin{bmatrix} t_1^1 & t_1^2 & t_1^3 \\ t_2^1 & t_2^2 & t_2^3 \\ t_3^1 & t_3^2 & t_3^3 \end{bmatrix}. \end{aligned} \tag{2.16}$$

Here $t^i_j = \underline{t}^i \cdot \underline{e}_j$ represents the j^{th} component of traction on i^{th} plane. Thus, if we want to write the stress matrix corresponding to ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system, its first column would be the representation of traction on plane whose normal is along the first coordinate axis (which is $\underline{e}_1$ here). Similarly, the second column will be the representation of traction on plane with normal along second coordinate axis and the third column will be the representation of traction on plane with normal along third coordinate axis. It is important to obtain the representation/ column form of the three tractions also in the same coordinate system. Usually, the following notation is used to write the stress matrix:

$$\left[\underline{\underline{\sigma}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}. \tag{2.17}$$

Basically, the off-diagonal elements denote shear components of traction and are denoted by τ whereas diagonal components denote normal components of traction and are denoted by σ. On comparing the above form with (2.16), τ_{ij} denotes the i^{th} component of traction on j^{th} plane.[3]

Cuboidal representation of stress tensor

Let us say we want to represent the stress tensor at a given point $\underline{x}$ in our body. Think of a cuboid around

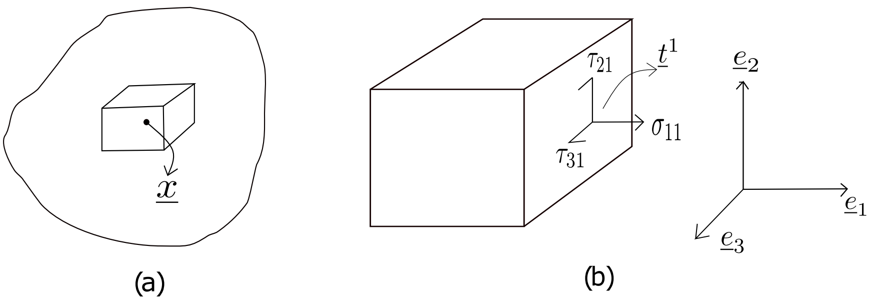


Figure 2.8: (a) Visualizing a cuboid centered at the point of interest $\underline{x}$ in our body (b) Zoomed view of the cuboid: components of traction that acts on its $\underline{e}_1$ face shown

point $\underline{x}$ with its center at $\underline{x}$ and its six faces chosen along $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, -$\underline{e}_1$, -$\underline{e}_2$ and -$\underline{e}_3$ (see Figure 2.8). The traction that acts on its $\underline{e}_1$ face is $\underline{t}^1$ whose three components are shown in Figure 2.8b. From the figure, it can be concluded that σ_{11} acts normal to the plane whereas τ_{21} and τ_{31} are in the plane. Thus, σ_{11} is the normal component of traction and τ_{21} and τ_{31} are the shear components of traction since they try to shear the body. To visualize shear, think of an internal section in the body. If the traction on this section has a non-zero component in the plane of the section, then the parts of the body on the two sides of the section will try to shear relative to each other, i.e., relatively displace along the plane of the section itself. Similarly, normal component of traction will lead to pushing or pulling between the two parts. If it is positive (negative), we call it tensile (compressive) since the two parts will try to pull

[3] In most books, τ_{ij} denotes the j^{th} component of traction on i^{th} plane but we will follow our convention since it comes out this way naturally. Our convention also aligns with the convention for components of other stress measures such as the first Piola-Kirchhoff stress tensor in nonlinear elasticity.

(push) themselves apart (into each other). Going back to our cuboid, the components of traction that acts on all its faces are shown in Figure 2.9. Remember that the second index denotes plane normal whereas the first index denotes direction. For faces with negative normals such as the bottom face, the

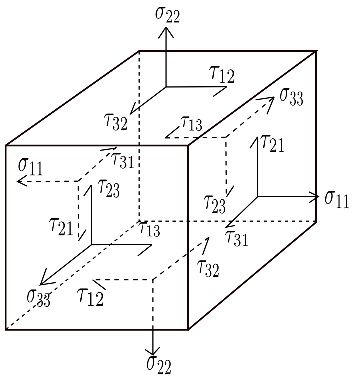


Figure 2.9: Traction components shown on all the six faces of the cuboid

plane normal is along -$\underline{e}_2$ and hence $\underline{t}^{-2}$ acts on it. We have also seen that

$$\underline{t}^{-2} = -\underline{t}^{2}.$$

So, the same traction components that act on the top face also act on the bottom face but directed oppositely. In fact, as per convention, a positive value for traction component implies that it acts along positive direction on positive planes but along negative direction on negative planes. Finally, if the various components drawn on the cuboid have to represent the components of stress matrix at a point $\underline{x}$ in the body, the cuboid must be shrunk to this point and made infinitesimally small so that all its faces pass through $\underline{x}$. In the above figures however, we have drawn the cuboid bigger and the point $\underline{x}$ at the center of the cuboid for better visualization.

2.5 Transformation of Stress matrix

We have learnt that stress matrix is the representation (matrix form) of stress tensor in a coordinate system. It changes from one coordinate system to another but the stress tensor itself remains unchanged. The goal here is to obtain a relationship between stress matrices in two different coordinate systems. The stress matrix in ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system is

$$\underline{\underline{[\sigma]}}_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \left[\begin{pmatrix} \\ \\ \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} \right] \text{ such that } \sigma_{ij} = \underline{t}^{j} \cdot \underline{e}_i.$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ [\underline{t}^1] & [\underline{t}^2] & [\underline{t}^3] \end{array}$$

Similarly, writing down the stress matrix in a new coordinate system $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$, we get

$$\left[\underline{\underline{\sigma}}\right]_{(\underline{\hat{e}}_1,\underline{\hat{e}}_2,\underline{\hat{e}}_3)} = \Big[\underbrace{\Big(\quad\Big)}_{[\underline{t}^{\hat{1}}]}\underbrace{\Big(\quad\Big)}_{[\underline{t}^{\hat{2}}]}\underbrace{\Big(\quad\Big)}_{[\underline{t}^{\hat{3}}]}\Big] \text{ such that } \hat{\sigma}_{ij} = \underline{t}^{\hat{j}} \cdot \underline{\hat{e}}_i \; (\neq \underline{t}^{\hat{j}} \cdot \underline{e}_i).$$

Using the above definition of $\hat{\sigma}_{ij}$, we can write

$$\hat{\sigma}_{ij} = \underline{t}^{\hat{j}} \cdot \underline{\hat{e}}_i = \left[\sum_{k=1}^{3} \underline{t}^k (\underline{\hat{e}}_j \cdot \underline{e}_k)\right] \cdot \underline{\hat{e}}_i = \sum_{k=1}^{3} (\underline{t}^k \cdot \underline{\hat{e}}_i)\,(\underline{\hat{e}}_j \cdot \underline{e}_k). \tag{2.18}$$

Now, we need to define a relationship between $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ and $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$. Let us think of the rotation tensor $\underline{\underline{R}}$ that relates one set of basis vectors to another set of basis vectors, i.e.,

$$\underline{\hat{e}}_i = \underline{\underline{R}}\, \underline{e}_i. \tag{2.19}$$

The matrix form of the above rotation tensor in $(\underline{e}_1,\ \underline{e}_2,\ \underline{e}_3)$ coordinate system, i.e., $\left[\underline{\underline{R}}\right]_{(\underline{e}_1,\ \underline{e}_2,\ \underline{e}_3)}$ will have its components defined by

$$R_{ij} = (\underline{\underline{R}}\underline{e}_j) \cdot \underline{e}_i = \underline{\hat{e}}_j \cdot \underline{e}_i. \tag{2.20}$$

Substituting it in equation (2.18), we get

$$\hat{\sigma}_{ij} = \sum_{k=1}^{3} (\underline{t}^k \cdot \underline{\hat{e}}_i)\,\underbrace{(\underline{\hat{e}}_j \cdot \underline{e}_k)}_{R_{kj}} = \sum_{k=1}^{3} \underbrace{\left(\sum_{m=1}^{3} t_m^k\, \underline{e}_m\right)}_{\underline{t}^k} \cdot \underline{\hat{e}}_i\, R_{kj} = \sum_k \sum_m t_m^k\, R_{mi} R_{kj}.$$

Now, by definition, $t_m^k = \sigma_{mk}$ because t_m^k represents the m^{th} component of traction $\underline{t}^k$ and will thus represent the m^{th} row in the k^{th} column of stress matrix. Hence

$$\hat{\sigma}_{ij} = \sum_k \sum_m \sigma_{mk} R_{mi} R_{kj} = \sum_k \sum_m R_{im}^T \sigma_{mk} R_{kj}.$$

Finally, writing this equation in a matrix form, we get

$$\left[\underline{\underline{\sigma}}\right]_{(\underline{\hat{e}}_1,\ \underline{\hat{e}}_2,\ \underline{\hat{e}}_3)} = \underbrace{\left[\underline{\underline{R}}\right]^T \left[\underline{\underline{\sigma}}\right] \left[\underline{\underline{R}}\right]}_{(\underline{e}_1,\ \underline{e}_2,\ \underline{e}_3)}. \tag{2.21}$$

This is the relation that we can use to transform a stress matrix from one coordinate system to another. In fact, the above relation holds for the transformation of matrix form of any second order tensor. One can use similar procedure to also relate the column form of a vector in two different coordinate systems. For example, one can deduce that

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{bmatrix} = \left[\underline{\underline{R}}\right]^T_{(\underline{e}_1,\ \underline{e}_2,\ \underline{e}_3)} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{2.22}$$

What is to be noticed here is that for transforming the vector components, we need to premultiply by $\left[\underline{\underline{R}}\right]^T$ whereas for relating basis vectors, we need to premultiply by $\underline{\underline{R}}$ (see equation (2.19)). This is because the vector $\underline{v}$ has to remain the same, we are just trying to write it in two different coordinate systems, i.e.,

$$\underline{v} = \sum v_i\, \underline{e}_i = \sum \hat{v}_i\, \underline{\hat{e}}_i.$$

Therefore, if the basis vector gets transformed with rotation $\underline{\underline{R}}$, the components would have to get transformed with $\left[\underline{\underline{R}}\right]^T$. Only then, $\underline{\underline{R}}^T\underline{\underline{R}}$ gets cancelled with each other to become $\underline{\underline{I}}$.

2.6 Solved examples

Q1. Show that $\underline{t}^n = \sum_i \underline{t}^i\,(\underline{n}\cdot\underline{e}_i) = \sum_i \underline{t}^{\hat{i}}\,(\underline{n}\cdot\underline{\hat{e}}_i)$, i.e, the formula is independent of what three planes are chosen to determine $\underline{t}^n$!

Solution: Think of $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ and $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ system which form the plane normals in the two cases. Let us expand the vector $\underline{e}_1$ in $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ basis, i.e., $\underline{e}_1 = \sum_{i=1}^{3} (\underline{e}_1\cdot\underline{\hat{e}}_i)\,\underline{\hat{e}}_i$. Here $(\underline{e}_1\cdot\underline{\hat{e}}_i)$ is the component of $\underline{e}_1$ along $\underline{\hat{e}}_i$. Generalizing this, we can write

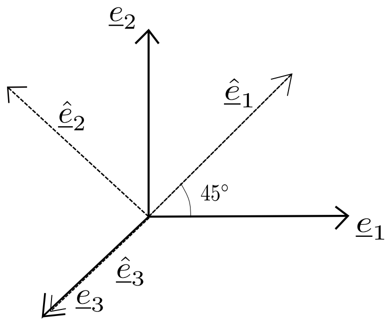
$$\underline{e}_j = \sum_i (\underline{e}_j\cdot\underline{\hat{e}}_i)\,\underline{\hat{e}}_i.$$

Now

$$\begin{aligned}
\underline{t}^n &= \sum_i \underline{t}^{\hat{i}}\,(\underline{n}\cdot\underline{\hat{e}}_i)\\
&= \sum_i \left(\sum_j \underline{t}^j\,(\underline{\hat{e}}_i\cdot\underline{e}_j)\right)(\underline{n}\cdot\underline{\hat{e}}_i)\ \ (\underline{t}^{\hat{i}} \text{ is expressed using tractions on } \underline{e}_1, \underline{e}_2, \text{ and } \underline{e}_3 \text{ planes})\\
&= \sum_j \underline{t}^j \sum_i (\underline{n}\cdot\underline{\hat{e}}_i)\,(\underline{\hat{e}}_i\cdot\underline{e}_j)\ \ (\text{upon changing the order of summation})\\
&= \sum_j \underline{t}^j \left(\underline{n}\cdot\sum_i \underline{\hat{e}}_i\,(\underline{\hat{e}}_i\cdot\underline{e}_j)\right) = \sum_j \underline{t}^j\,(\underline{n}\cdot\underline{e}_j) \quad \left(\because \underline{e}_j = \sum_i \underline{\hat{e}}_i\,(\underline{\hat{e}}_i\cdot\underline{e}_j) \text{ as derived earlier}\right).
\end{aligned}$$

NOTE: This also proves that stress tensor is independent of what three planes are chosen to form it, i.e., $\underline{\underline{\sigma}} = \sum_i \underline{t}^i \otimes \underline{e}_i = \sum_i \underline{t}^{\hat{i}} \otimes \underline{\hat{e}}_i$.

Q2. Suppose $[\underline{t}^1] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $[\underline{t}^2] = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$, $[\underline{t}^3] = \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix}$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system.
What will be the traction on a plane with normal $\underline{n} = \hat{\underline{e}}_1$ where $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$ is obtained from rotation of $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ about $\underline{e}_3$ by 45° (see the figure below)? What are the normal and shear components of traction on this plane?



Solution: We simply have to use the formula

$$\underline{t}^n = \sum_i \underline{t}^i \, (\underline{n} \cdot \underline{e}_i).$$

Keep in mind that the above is a tensor formula. To use it, we must write every quantity involved in the above formula in the same coordinate system! Let us use $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system here:

$$\Rightarrow [\underline{n}]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} = [\hat{\underline{e}}_1]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

Therefore

$$[\underline{t}^n] = \sum_i [\underline{t}^i] \, ([\underline{n}] \cdot [\underline{e}_i]) = 1/\sqrt{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1/\sqrt{2} \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix} = 1/\sqrt{2} \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} .$$

CAUTION: If you use $[\underline{n}] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, you will get wrong result! Never mix the coordinate system! Either use $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ for all or use $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$ for all. The final result for $\underline{t}^n$ will be in the coordinate system that you choose. In order to get the normal component of traction, we need to find $\underline{t}^n \cdot \underline{n}$ or $\underline{t}^{\hat{1}} \cdot \hat{\underline{e}}_1$ which when expressed in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system yields

$$1/\sqrt{2} \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{7}{2}.$$

One can likewise obtain shear components by taking the dot product of traction vector with $\underline{\hat{e}}_2$ and $\underline{\hat{e}}_3$.

Q3. Show that the component of a traction vector on $\underline{n}$-plane in the direction $\underline{m}$ equals the component of the traction on $\underline{m}$-plane in the direction $\underline{n}$, i.e, $\underline{t}^n \cdot \underline{m} = \underline{t}^m \cdot \underline{n}$.

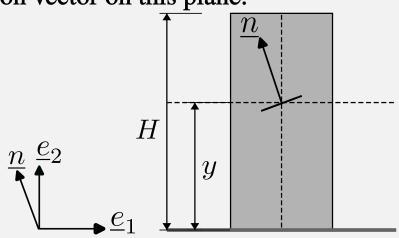
Solution: To prove this, we will use the definition of stress tensor as follows:

$$
\begin{aligned}
\underline{t}^n \cdot \underline{m} &= \left(\underline{\underline{\sigma}}\,\underline{n}\right) \cdot \underline{m} \\
&= \underline{n} \cdot \left(\underline{\underline{\sigma}}^T\,\underline{m}\right) \\
&= \underline{n} \cdot \left(\underline{\underline{\sigma}}\,\underline{m}\right) \;\; \text{(due to symmetry of } \underline{\underline{\sigma}} \text{ which will be proved later)} \\
&= \underline{n} \cdot \underline{t}^m \;.
\end{aligned}
$$

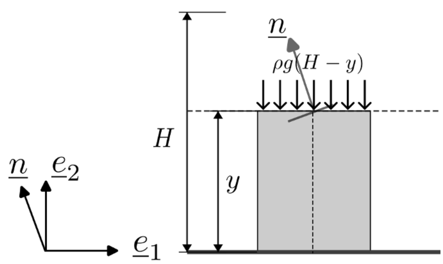
Q4. Consider a vertical bar having mass density ρ. Assume its length be to H and is subjected to uniform body force due to gravity. Find the traction vector on an infinitesimal internal section of the bar located at the center of its cross-section with outward normal

$$\underline{n} = -\sin\theta\;\underline{e}_1 + \cos\theta\;\underline{e}_2$$

and at a height of y from the base (see figure). Also find the normal and tangential components of the traction vector on this plane.



Solution:



Taking a horizontal section at height y and writing force balance of top part of bar yields

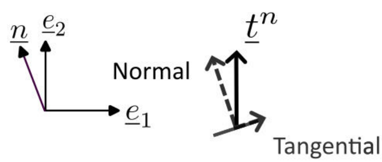
$$\left[\underline{t}^2\right] = \begin{bmatrix} 0 \\ -\rho g(H-y) \\ 0 \end{bmatrix} .$$

Similarly taking a vertical section all along the bar and doing force balance either of left or right portion of the bar yields

$$[\underline{t}^1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

One can likewise show that $[\underline{t}^3]$ also vanishes although it turns out this information is not required. In order to obtain traction on an infinitesimal inclined plane at the center of the horizontal section of the bar, we can use the tetrahedron formula, i.e.,

$$\begin{aligned}[\underline{t}^n] &= \sum_i [\underline{t}^i]\left([\underline{n}]\cdot[\underline{e}_i]\right)\\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}(-\sin\theta) + \begin{bmatrix} 0 \\ -\rho g(H-y) \\ 0 \end{bmatrix}(\cos\theta) + [\underline{t}^3]\ 0 = \begin{bmatrix} 0 \\ -\rho g(H-y)\cos\theta \\ 0 \end{bmatrix}.\end{aligned}$$



The normal component of traction vector is given by the projection of $\underline{t}^n$ along $\underline{n}$:

$$t_{\text{normal}} = \underline{t}^n \cdot \underline{n} = -\rho g(H-y)\cos\theta\ \ (\underline{e}_2 \cdot \underline{n}) = -\rho g(H-y)\cos^2\theta.$$

The shear or tangential component of traction can then be obtained as follows:

$$t_{\text{tangential}} = \underline{t}^n \cdot \underline{n}^{\perp} = -\rho g(H-y)\cos\theta\ \underline{e}_2 \cdot (\cos\theta\ \underline{e}_1 + \sin\theta\ \underline{e}_2) = -\rho g(H-y)\cos\theta\sin\theta$$

where $\underline{n}^{\perp}$ is a unit vector perpendicular to $\underline{n}$ and lies in the plane of inclined section.

Q5. Suppose the stress matrix in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system is given by

$$[\underline{\underline{\sigma}}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Think of a new coordinate system which is obtained by rotation of $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system by 45° about $\underline{e}_3$ as in problem 2 above. Find the stress matrix in the new coordinate system.

First, we need to find the rotation matrix. As derived earlier, this rotation matrix will be

$$[\underline{\underline{R}}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \cos(45°) & -\sin(45°) & 0 \\ \sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$