Department of Mathematics

MTL 145: Number Theory

Problems on Divisibility

These problems are taken from *An Introduction to the Theory of Numbers* **by** Niven, Zuckerman & Montgomery **with minor changes.**

1. Find all integers x, y, z such that $6x + 10y + 15z = 1$.

2. Prove that the product of three consecutive integers is divisible by 6, and of four consecutive integers by 24. State and prove the generalization to products of n consecutive integers.

3. Prove that $2 \mid (n^2 - n)$, $6 \mid (n^3 - n)$, and $30 \mid (n^5 - n)$ for all $n \in \mathbb{Z}$. Can you generalize this?

4. Let s and $g > 0$ be given integers. Prove that there exist integers x, y such that $x + y = s$ and $(x, y) = g$ if and only if $g \mid s$.

5. Let g and ℓ be given positive integers. Prove that there exist integers x, y such that $(x, y) = g$ and $[x, y] = \ell$ if and only if $g \mid \ell$.

6. Let c and $g > 0$ be given integers. Prove that there exist integers x, y such that $xy = c$ and $(x, y) = g$ if and only if $g^2 \mid c$.

7. Find all triples of positive integers a, b, c satisfying $(a, b, c) = 10$ and $[a, b, c] = 100$ simultaneously.

8. Let $n \geq 2$ and k be any positive integers. Prove that $(n - 1) \mid (n^k - 1)$. More generally, if x, y, k are integers with $x \neq y$ and $k \geq 1$, then $(x - y) \mid (x^k - y^k)$.

9. Let $n \geq 2$ and k be any positive integers. Prove that $(n-1)^2 \mid (n^k-1)$ if and only if $(n-1) \mid k$.

10. Prove that $(a, b) = (a, b, ax + by)$ for all integers x, y.

11. Prove that any positive integer n can be *uniquely* expressed in the form

$$n = 3^m + a_{m-1}3^{m-1} + a_{m-2}3^{m-2} + \cdots + a_0$$

where each $a_i \in \{-1, 0, 1\}$.

12. Prove that there are no positive integers $a, b, n > 1$ such that $(a^n - b^n) \mid (a^n + b^n)$.

13. If a and $b > 2$ are any positive integers, prove that $2^a + 1$ is not divisible by $2^b - 1$.

14. Prove that if $m > n$, then $(a^{2^n} + 1) \mid (a^{2^m} - 1)$. Show that if a, m, n are positive integers with $m \neq n$, then

$$(a^{2^m} + 1, a^{2^n} + 1) = \begin{cases} 1 & \text{if } a \text{ is even;} \\ 2 & \text{if } a \text{ is odd.} \end{cases}$$

Deduce that there are infinitely many primes.

15. Show that if $(a, b) = 1$ and p is an odd prime, then $(a + b, \frac{a^p+b^p}{a+b}) = 1$ or p.

16. For $n \geq 1$, suppose $2^n + 1 = ab$ for some integers a, b each greater than 1. Show that $2^m \mid (a - 1)$ if and only if $2^m \mid (b - 1)$.

17. Show that $((n + 1)! + 1, n! + 1) = 1$ for each positive integer n.