Department of Mathematics

MAL 145: Number Theory

Problems on Primes

These problems are taken from *An Introduction to the Theory of Numbers* **by** Niven, Zuckerman & Montgomery **with minor changes.**

1. Starting with any positive integer n, subtract double the units digit from the integer obtained from n by removing the units digit, giving a smaller number r. For example, if $n = 41283$ with units digit 3 we subtract 6 from 4128 to get $r = 4122$. Prove that $7 \mid n$ if and only if $7 \mid r$. Devise a similar test of divisibility for 13.

2. Prove that any positive integer of the form $3k + 2$ or $4k + 3$ or $6k + 5$ has a prime factor of the same form.

3. If $(x, 2) = (y, 2) = 1$ or if $(x, 3) = (y, 3) = 1$, prove that $x^2 + y^2$ cannot be a perfect square.

4. Show that there is a one-to-one correspondence between twin primes and numbers n such that $n^2 - 1$ has exactly four positive divisors.

5. Prove that $(a, b, c) \cdot [a, b, c] = abc$ implies $(a, b) = (b, c) = (a, c) = 1$.

6. Prove that $[a, b, c] \cdot (ab, bc, ac) = |abc|$.

7. Given integers a, b, c, d, m, n, u, v satisfying $ad - bc = \pm 1$, $u = am + bn$, $v = cm + dn$, prove that $(m, n) = (u, v)$.

8. Prove that there are infinitely many primes of the form $4n+3$ and of the form $6n+5$. Is this also true for numbers of the form $8n + 7$?

9. Show that $n \mid (n - 1)!$ for all composite $n > 4$.

10. Suppose $n > 1$. Show that the sum of the positive integers not exceeding n divides the product of the positive integers not exceeding n if and only if $n + 1$ is composite.

11. Suppose m, n are integers > 1, and that $\frac{\log m}{\log n} = \frac{a}{b}$. Prove that there exists an integer c such that $m = c^a$, $n = c^b$.

12. Prove that no polynomial $f(x)$ of degree > 1 with integral coefficients can represent a prime for every positive integer x.

13. Show that if $m^4 + 4^n$ is prime, then m is odd and n is even, except when $m = n = 1$.

14. Prove that
$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + \tfrac{1}{1999} - \tfrac{1}{2000} = \tfrac{1}{1001} + \tfrac{1}{1002} + \cdots + \tfrac{1}{2000}.$$

15. Prove that in any block of consecutive positive integers there is a *unique* integer divisible by a higher power of 2 than any of the others. Hence prove that there is no integer among the 2^{n+1} numbers
$$\pm\tfrac{1}{k} \pm \tfrac{1}{k+1} \pm \cdots \pm \tfrac{1}{k+n}$$
where all possible combinations of plus and minus signs are allowed and where n, k are positive integers.

16. Prove that an odd integer $n > 1$ is a prime if and only if it is *not* expressible as a sum of three or more consecutive positive integers.

17. If $2^n + 1$ is a prime for some integer n, then prove that $n = 2^k$ for some positive integer k.

18. If $2^n - 1$ is a prime for some integer n, then prove that n is prime.

19. Let $k \geq 3$. Find all sets $a_1, \ldots, a_k$ of positive integers such that the sum of any triplet is divisible by each member of the triplet.

20. Let a, b, c, d be integers. If g is a divisor of each of $ab, cd, ac+bd$, prove that it is also a divisor of ac and bd.

21. Show that 24 is the largest integer divisible by all integers less than its square root.