Semigroup Forum (2026) 113:465–481
https://doi.org/10.1007/s00233-026-10658-y

RESEARCH ARTICLE

On quotients of two-dimensional numerical semigroups that are ordinary

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Received: 21 January 2026 / Accepted: 22 June 2026 / Published online: 29 July 2026
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Abstract
A numerical semigroup S is ordinary if its gap set $\mathrm{G}(S)$ is either empty or of the form $\{1, \ldots, n\}$ for some positive integer n. For a positive integer d, by the quotient of S by d we mean the numerical semigroup $\frac{1}{d}S = \{n \in \mathbb{Z}_{\geq 0} : nd \in S\}$. For coprime positive integers a, b, we investigate values of d for which the numerical semigroup $\frac{1}{d}\langle a, b\rangle$ is ordinary.

Keywords Numerical semigroup · Embedding dimension · Quotient of a numerical semigroup · Ordinary numerical semigroup

1 Introduction

By $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ we mean the set of non-negative integers and the set of positive integers, respectively. A numerical semigroup is a subset S of $\mathbb{Z}_{\geq 0}$ that is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is a finite set. For $a_1, \ldots, a_k \in \mathbb{N}$, let

$$\langle a_1, \ldots, a_k\rangle = \left\{a_1x_1 + \cdots + a_kx_k : x_i \in \mathbb{Z}_{\geq 0}\right\}.$$

Then $\langle a_1, \ldots, a_k\rangle$ is a numerical semigroup if and only if $\gcd(a_1, \ldots, a_k) = 1$.

The set $A = \{a_1, \ldots, a_k\}$ is called a system of generators of the semigroup $S = \langle a_1, \ldots, a_k\rangle$. For a semigroup S, A is a minimal system of generators if A generates S and no proper subset of A generates S. In fact, if $S^\star = S \setminus \{0\}$, then every system of generators contains $S^\star \setminus (S^\star + S^\star)$ making this the unique system of minimal

Communicated by Nathan Kaplan.

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generators. This system of generators is finite, and the cardinality of this minimal system of generators is called the embedding dimension of S, and is denoted by $\mathrm{e}(S)$.

Given a numerical semigroup S, we define the set $\mathrm{G}(S)$ of gaps of S to be $\mathbb{N} \setminus S$. The largest element in $\mathrm{G}(S)$ is the Frobenius number $\mathrm{F}(S)$ and the cardinality of $\mathrm{G}(S)$ is the genus $\mathrm{g}(S)$ of the numerical semigroup S. The multiplicity of S is the least positive element in S, and is denoted by $\mathrm{m}(S)$.

An element $n \in \mathrm{G}(S)$ is said to be a fundamental gap if $kn \in S$ for each $k > 1$. It is easy to see that the set of fundamental gaps $\mathrm{FG}(S)$ determines the numerical semigroup S uniquely, as any element in $\mathrm{G}(S)$ is just a divisor of some fundamental gap. The set $\mathrm{FG}(S)$ of all fundamental gaps of S has been determined by Rosales [3] in the case where S has embedding dimension 2.

Given a numerical semigroup S and a positive integer d, the quotient of S by d is the set

$$\frac{1}{d}S = \{n \in \mathbb{Z}_{\geq 0} : nd \in S\}.$$

The following results are basic to the study of quotient of S by d. Proposition 1.1 and Proposition 1.2 characterize d for which the complement of the quotient of S by d is either the empty set or the singleton $\{1\}$. Proposition 1.3 shows that in the case $S = \langle a, b\rangle$, it is no loss of generality to assume that d may be chosen to be coprime to both a and b.

Proposition 1.1 ([4, Proposition 6.1]) *If S is a numerical semigroup and d a positive integer, then the set $\frac{1}{d}S$ is a numerical semigroup that contains S. Moreover, $\frac{1}{d}S = \mathbb{Z}_{\geq 0}$ if and only if $d \in S$.*

Proposition 1.2 ([4, Corollary 6.3]) *If S is a numerical semigroup and d a positive integer, then $\frac{1}{d}S = \mathbb{Z}_{\geq 0} \setminus \{1\}$ if and only if $d \in FG(S)$. In particular, $\frac{1}{d}S = \mathbb{Z}_{\geq 0} \setminus \{1\}$ for $d = F(S)$.*

Proposition 1.3 ([6, Lemma 14]) *Let a, b, d, k be positive integers with* $\gcd(a, kb) = 1$. *Then*

$$\frac{1}{kd}\langle a, kb\rangle = \frac{1}{d}\langle a, b\rangle.$$

A numerical semigroup S is said to be *ordinary* if $\mathrm{G}(S) = \emptyset$ or $\{1, \ldots, n\}$ for some positive integer n. In this paper, we investigate which quotients of numerical semigroups by positive integers are ordinary numerical semigroups. If $d \in S$, then it is apparent that $\frac{1}{d}S = \mathbb{Z}_{\geq 0}$, so that quotients by elements of the semigroup are always ordinary. Therefore, we henceforth assume $d \in \mathrm{G}(S)$ and ask to characterize d for which $\frac{1}{d}S$ is an ordinary numerical semigroup. Since $\mathrm{G}\left(\frac{1}{d}S\right) = \{1\}$ for $d = \mathrm{F}(S)$, we see that $d = \mathrm{F}(S)$ provides the largest instance of this. We investigate this problem in the case $S = \langle a, b\rangle$, where $\gcd(a, b) = 1$ and $d \notin \langle a, b\rangle$.

In Sect. 2, we provide some preliminary results for the rest of the paper for the case where $S = \langle a, b\rangle$. In particular, given positive and coprime integers a, b, we provide a characterization of positive integers d for which $\frac{1}{d}\langle a, b\rangle$ is an ordinary numerical semigroup. In Sect. 3 we determine the smallest value of d for which $\frac{1}{d}\langle a, b\rangle$ is an ordinary numerical semigroup for all choices of a, b. In Sect. 4 we provide an estimate for the number of values of d for which $\frac{1}{d}\langle a, b\rangle$ is an ordinary numerical semigroup for all choices of a, b.

2 Preliminary results

In this section, we provide some results essential to our investigations in this paper. The main result is the connection between ordinary numerical semigroups and fundamental gaps of a specific order; see Lemma 2.6.

Lemma 2.1 *If a, b are positive integers with $\gcd(a,b) = 1$, then $n \in \langle a, b\rangle$ if and only if $\left\lfloor \frac{n}{b} \right\rfloor \geq nb^{-1} \pmod a$.*

Proof Suppose $n \equiv bs \pmod a$, $0 \leq s \leq a-1$. Then $n = bs + at \in \langle a, b\rangle$ if and only if $t \in \mathbb{Z}_{\geq 0}$. Now $n \equiv bs \pmod a$ is equivalent to $s \equiv nb^{-1} \pmod a$, and so $n \in \langle a, b\rangle$ if and only if $n \geq bs$, which is equivalent to $\frac{n}{b} \geq nb^{-1} \pmod a$. □

Proposition 2.2 *Suppose a, b, d are positive integers with $\gcd(a,b) = 1$ and $d \notin \langle a, b\rangle$. If $d = bs - ar$ with $1 \leq s \leq a-1$ and $r \geq 1$, then $\left\{1, \ldots, \left\lfloor \frac{a-1}{s} \right\rfloor\right\} \subseteq G\left(\frac{1}{d}\langle a, b\rangle\right)$.*

Proof Suppose $n \in \left\{1, \ldots, \left\lfloor \frac{a-1}{s} \right\rfloor\right\}$ and $n \in \frac{1}{d}\langle a, b\rangle$. Then $nd \in \langle a, b\rangle$, and so $\left\lfloor \frac{nd}{b} \right\rfloor \geq (nd)b^{-1} \pmod a$ by Lemma 2.1. But $\left\lfloor \frac{nd}{b} \right\rfloor < ns = (nd)b^{-1} \pmod a$, and this contradiction proves that each $n \in \left\{1, \ldots, \left\lfloor \frac{a-1}{s} \right\rfloor\right\}$ is in $\mathrm{G}\left(\frac{1}{d}\langle a, b\rangle\right)$. □

Notation 2.3 Let S be a numerical semigroup. Then

$$\mathrm{ORD}(S) = \left\{d \in \mathrm{G}(S) : \tfrac{1}{d}S \text{ is an ordinary numerical semigroup}\right\}.$$

As observed in Sect. 1, $\max \mathrm{ORD}(S) = \mathrm{F}(S)$ for every numerical semigroup S.

The notion of fundamental gaps in numerical semigroups was extended by Elizeche and Tripathi in [2]. Given a positive integer λ and a numerical semigroup S, we call a positive integer n a *fundamental gap of S of order λ* if $kn \notin S$ for each $k \in \{1, \ldots, \lambda\}$ and $kn \in S$ for each integer $k > \lambda$. The following result describes the set of fundamental gaps of order λ in two-dimensional numerical semigroups.

Theorem 2.4 ([2, Theorem 1]) *Let $S = \langle a, b\rangle$, where $\gcd(a,b) = 1$. The set of fundamental gaps of order λ is given by*

$$FG(S, \lambda) = A \cup B \cup C \cup D \cup E,$$

where

$$\begin{aligned}
A &= \left\{bs - ar : \tfrac{(\lambda-1)b}{\lambda} < r \leq \tfrac{\lambda b}{\lambda+1}, \tfrac{2\lambda a}{2\lambda+1} \leq s \leq a-1\right\},\\
B &= \left\{bs - ar : \tfrac{(\lambda-1)b}{\lambda} < r \leq \tfrac{(2\lambda-1)b}{2\lambda+1}, \tfrac{(2\lambda-1)a}{2\lambda} \leq s < \tfrac{2\lambda a}{2\lambda+1}\right\},\\
C &= \left\{bs - ar : 1 \leq r \leq \tfrac{b}{2\lambda+1}, s = \tfrac{a}{\lambda+1}\right\},\\
D &= \left\{bs - ar : 1 \leq r \leq \tfrac{b}{2\lambda}, \tfrac{a}{\lambda+1} < s < \tfrac{a}{\lambda}, 0 < a \bmod s \leq \tfrac{s}{2}\right\},\\
E &= \left\{bs - ar : 1 \leq r \leq \tfrac{b}{2\lambda+1}, \tfrac{a}{\lambda+1} < s < \tfrac{a}{\lambda}, a \bmod s > \tfrac{s}{2}\right\}.
\end{aligned}$$

Remark 2.5 Observe that the sets C, D, E are pairwise disjoint, and that when $\lambda = 1$, $A = D$ and $B = C \cup E$.

The following result makes the connection between ordinary numerical semigroups and fundamental gaps of a specific order.

Lemma 2.6 *Let S be a numerical semigroup and d a positive integer. Then $d \in ORD(S)$ if and only if d is a fundamental gap of S of order $F\left(\frac{1}{d}S\right)$.*

Proof If $d \in \mathrm{ORD}(S)$, then $\mathrm{G}\left(\frac{1}{d}S\right) = \{1, \ldots, n\}$, where $n = \mathrm{F}\left(\frac{1}{d}S\right)$. Thus, $kd \notin S$ for each $k \in \{1, \ldots, n\}$ and $kd \in S$ for each $k > n$, implying that d is a fundamental gap of order n.

Conversely, suppose d is a fundamental gap of order $\mathrm{F}(S')$, where $S' = \frac{1}{d}S$. Thus, $kd \notin S$ for each $k \in \{1, \ldots, \mathrm{F}(S')\}$ and $kd \in S$ for each $k > \mathrm{F}(S')$, so that $\mathrm{G}(S') = \{1, \ldots, \mathrm{F}(S')\}$. □

By Proposition 2.2, the gap set of $\frac{1}{d}\langle a, b\rangle$ must contain $\left\{1, \ldots, \left\lfloor \frac{a-1}{s} \right\rfloor\right\}$. Theorem 2.4 and Lemma 2.6 help characterize a, b, d for which the gap set is equal to $\left\{1, \ldots, \left\lfloor \frac{a-1}{s} \right\rfloor\right\}$.

Proposition 2.7 *Suppose a, b, d are positive integers with $\gcd(a, b) = 1$ and $d \notin \langle a, b\rangle$. If $d = bs - ar$ with $1 \le s \le a - 1$ and $r \ge 1$, then*

$$G\left(\tfrac{1}{d}\langle a, b\rangle\right) = \left\{1, \ldots, \left\lfloor \tfrac{a-1}{s} \right\rfloor\right\} \quad \text{if and only if} \quad d \in C \cup D \cup E,$$

where the sets C, D, E are as defined in Theorem 2.4.

Proof We use the notations of Theorem 2.4. Let $d = bs - ar$ with $1 \le s \le a - 1$ and $r \ge 1$, and let $\lambda = \lfloor \frac{a-1}{s} \rfloor$. Applying Lemma 2.6 to $S = \langle a, b\rangle$ and by Theorem 2.4, we have

$$\mathrm{G}\left(\tfrac{1}{d}\langle a, b\rangle\right) = \{1, \ldots, \lambda\} \quad \text{if and only if} \quad d \in \mathrm{F}G(S, \lambda) = A \cup B \cup C \cup D \cup E.$$

If $\lambda = 1$, $A = D$ and $B = C \cup E$ by Remark 2.5, so $d \in C \cup D \cup E$.
If $\lambda \ge 2$, then $\frac{2\lambda}{2\lambda+1} > \frac{2\lambda-1}{2\lambda} \ge \frac{3}{4} > \frac{s}{a}$ since $f(x) = x/(x+1)$ is increasing in $(0, \infty)$ and since $a - 1 \ge 2s$. Therefore $d \notin A \cup B$, and so $d \in C \cup D \cup E$. □

3 Evaluation of the minimum element in ORD($\langle a, b\rangle$)

In this section, we give a characterization of $d \in \mathrm{ORD}(\langle a, b\rangle)$ in Theorem 3.1, and use this to determine the minimum element in $\mathrm{ORD}(\langle a, b\rangle)$ for all choices of a, b in Theorem 3.6.

Theorem 3.1 *Let a, b, d be positive integers such that $\gcd(a, b) = 1$ and $d \notin \langle a, b\rangle$. If $d = bs - ar$, $1 \le s \le a - 1$ and $1 \le r < bs/a$, then $d \in ORD(\langle a, b\rangle)$ if and only if one of the following holds.*

(I) $s \leq \frac{a}{2}$ *and*

$$s \mid a, \quad r \leq \frac{b}{2\frac{a}{s}-1}, \tag{1a}$$

$$\text{or } 0 < a \pmod{s} \leq \frac{s}{2}, \quad r \leq \frac{b}{2\left\lfloor \frac{a}{s} \right\rfloor}, \tag{1b}$$

$$\text{or } a \pmod{s} > \frac{s}{2}, \quad r \leq \frac{b}{2\left\lfloor \frac{a}{s} \right\rfloor + 1}. \tag{1c}$$

(II) $s \geq \frac{a}{2}$ *and*

$$r \leq \frac{\lfloor \mu \rfloor b}{\lfloor \mu \rfloor + 1}, \quad \lfloor \mu \rfloor = \left\lfloor \mu + \tfrac{1}{2} \right\rfloor, \tag{2a}$$

$$\text{or } r \leq \frac{\lfloor 2\mu \rfloor b}{\lfloor 2\mu \rfloor + 2}, \quad \lfloor \mu \rfloor \neq \left\lfloor \mu + \tfrac{1}{2} \right\rfloor, \tag{2b}$$

where $\mu = \frac{s}{2(a-s)}$.

Proof From Lemma 2.6, we have that $\frac{1}{d}\langle a, b\rangle$ is an ordinary numerical semigroup if and only if d is a fundamental gap of order λ for some value of λ. Then $d \in \mathrm{FG}(S, \lambda) = A \cup B \cup C \cup D \cup E$ as defined in Theorem 2.4. Let $d = bs - ar$.

Let $d \in A$. Then $\frac{(\lambda-1)b}{\lambda} < r \leq \frac{\lambda b}{\lambda+1}$, $\frac{2\lambda a}{2\lambda+1} \leq s \leq a-1$. The second inequality implies that $\lambda \leq \frac{s}{2(a-s)}$. In this case, we have that $\frac{(\lambda-1)b}{\lambda} < r \leq \frac{\lambda b}{\lambda+1}$, for each λ that satisfies the inequality.

Let $d \in B$. Then $\frac{(\lambda-1)b}{\lambda} < r \leq \frac{(2\lambda-1)b}{2\lambda+1}$, $\frac{(2\lambda-1)a}{2\lambda} \leq s < \frac{2\lambda a}{2\lambda+1}$ In this case we have $\frac{s}{2(a-s)} < \lambda \leq \frac{a}{2(a-s)} = \frac{s}{2(a-s)} + \frac{1}{2}$.

Putting the last two cases together, we have for $\lambda \leq \mu = \frac{s}{2(a-s)}$, that $\frac{(\lambda-1)b}{\lambda} < r \leq \frac{\lambda b}{\lambda+1}$. Note that if we increase λ to $\lambda+1$, the intervals overlap exactly. So we can write this simply as $r \leq \frac{\lfloor \mu \rfloor b}{\lfloor \mu \rfloor + 1}$. Now, if we can find λ such that $\mu < \lambda \leq \mu + \frac{1}{2}$, we have the extra interval $\frac{((\lfloor \mu \rfloor+1)-1)b}{\lfloor \mu \rfloor+1} = \frac{\lfloor \mu \rfloor b}{\lfloor \mu \rfloor+1} < r \leq \frac{(2(\lfloor \mu \rfloor+1)-1)b}{2(\lfloor \mu \rfloor+1)+1} = \frac{(2\lfloor \mu \rfloor+1)b}{2\lfloor \mu \rfloor+3}$ since $\lambda = \lfloor \mu \rfloor + 1$. This finishes case (II) in our theorem.

Let $d \in C$. Then $1 \leq r \leq \frac{b}{2\lambda+1}$ and $s = \frac{a}{\lambda+1}$. The second equation implies $\lambda = \frac{a-s}{s}$ with $s \mid a$. Therefore $1 \leq r \leq \frac{b}{2\frac{a-s}{s}+1} = \frac{b}{2\frac{a}{s}-1}$.

Let $d \in D$. Then $1 \leq r \leq \frac{b}{2\lambda}$, $\frac{a}{\lambda+1} < s < \frac{a}{\lambda}$, $0 < a \pmod{s} \leq \frac{s}{2}$. From the second inequality we have $\frac{a}{s} - 1 < \lambda < \frac{a}{s}$. Then $\lambda = \lfloor \frac{a}{s} \rfloor$. Substituting in the first inequality we get $1 \leq r \leq \frac{b}{2\lfloor \frac{a}{s} \rfloor}$.

Let $d \in E$. Then $1 \leq r \leq \frac{b}{2\lambda+1}$, $\frac{a}{\lambda+1} < s < \frac{a}{\lambda}, a \pmod{s} > \frac{s}{2}$. Again, from the second inequality, we have $\lambda = \lfloor \frac{a}{s} \rfloor$. Thus $r \leq b/\left(2\lfloor \frac{a}{s} \rfloor + 1\right)$. This finishes case (I) in our theorem. □

Proposition 3.2 *Let a, b, d be positive integers such that $\gcd(a, b) = 1$ and $d \notin \langle a, b\rangle$. Let $d = bs - ar$, $1 \le s \le a - 1$ and $1 \le r < bs/a$. If*

$$d < (a - s)r,$$

then $d \notin ORD(\langle a, b\rangle)$.

Proof By Theorem 3.1, when $s \le a/2$, $d \notin \mathrm{ORD}(\langle a, b\rangle)$ provided r multiplied by $2\frac{a}{s} - 1$, by $2\lfloor\frac{a}{s}\rfloor$, and by $2\lfloor\frac{a}{s}\rfloor + 1$ exceeds b, in the three cases given by eqns. (1a), (1b), and (1c), respectively. Similarly, when $s \ge a/2$, $d \notin \mathrm{ORD}(\langle a, b\rangle)$ provided r multiplied by $(\lfloor\mu\rfloor + 1)/\lfloor\mu\rfloor$ or by $(\lfloor 2\mu\rfloor + 2)/\lfloor 2\mu\rfloor$ exceeds b, in the two cases given by eqns. (2a) & (2b), respectively.

Since in the cases given by eqn. (1b), $\lfloor\frac{a}{s}\rfloor \ge \frac{a}{s} - \frac{1}{2}$ and in cases given by eqn. (1c), $\lfloor\frac{a}{s}\rfloor \ge \frac{a}{s} - 1$, we have $d \notin \mathrm{ORD}(\langle a, b\rangle)$ provided $r\left(2\frac{a}{s} - 1\right) > b$. Again, since $f(x) = x/(x + 1)$ is increasing in $(0, \infty)$, $\lfloor\mu\rfloor/(\lfloor\mu\rfloor + 1)$ and $\lfloor 2\mu\rfloor/(\lfloor 2\mu\rfloor + 2)$ are both bounded above by $\mu/(\mu + 1)$. So $d \notin \mathrm{ORD}(\langle a, b\rangle)$ provided $(\mu + 1)r > \mu b$.

Replace r by $(bs - d)/a$ in each to get

$$b < \frac{bs-d}{a}\left(2\frac{a}{s} - 1\right) = 2b - \frac{bs}{a} - \frac{2d}{s} + \frac{d}{a}$$

in the first case, and get

$$\frac{bs-d}{a}\left(\frac{s}{a-s} + 2\right) > \frac{bs}{a-s}$$

in the second case. The first case reduces to $abs - bs^2 - 2ad + ds > 0$, or to $(a - s)(bs - d) > ad$. The second case reduces to $(bs - d)(2a - s) > abs$, or to $(bs - d)(a - s) > ad$. Both cases are restatements of $d < (a - s)r$. □

Corollary 3.3 *If b is an odd integer, $b > 1$, then*

$$ORD(\langle 2, b\rangle) = \left\{b - 2r : 1 \le r \le \tfrac{b}{3}\right\}.$$

Proof Let $d \in \mathrm{ORD}(\langle 2, b\rangle)$, so that $d = b - 2r$ with $1 \le r \le (b - 1)/2$. By Theorem 3.1, the only case that applies is given by eqn. (1a), and in this case $r \le \frac{b}{3}$. □

Corollary 3.4 *If b is an integer such that $3 \nmid b$ and $b > 3$, then*

$$ORD(\langle 3, b\rangle) = \left\{b - 3r : 1 \le r \le \tfrac{b}{5}\right\} \bigcup \left\{2b - 3r : 1 \le r \le \tfrac{b}{2}\right\}.$$

Proof Let $d \in \mathrm{ORD}(\langle 3, b\rangle)$, so that $d = bs - 3r$ with $s \in \{1, 2\}$ and $1 \le r < bs/3$. If $s = 1$, by Theorem 3.1, the only case that applies is given by eqn. (1a), and in this case $r \le \frac{b}{5}$. If $s = 2$, by Theorem 3.1, the only case that applies is given by eqn. (2a), and in this case $r \le \frac{b}{2}$. Note that the two sets are disjoint since $b - 3r_1 = 2b - 3r_2$ implies $3 \mid b$. □

Corollary 3.5 *If b is an odd integer and $b > 4$, then*

$$ORD(\langle 4, b\rangle)$$
$$= \left\{b - 4r : 1 \le r \le \tfrac{b}{7}\right\} \bigcup \left\{2b - 4r : 1 \le r \le \tfrac{b}{3}\right\} \bigcup \left\{3b - 4r : 1 \le r \le \tfrac{3b}{5}\right\}.$$

Proof Let $d \in \mathrm{ORD}(\langle 4, b\rangle)$, so that $d = bs - 4r$ with $s \in \{1, 2, 3\}$ and $1 \le r < bs/4$. If $s \in \{1, 2\}$, by Theorem 3.1, the only case that applies is given by eqn. (1a), and in this case $r \le \frac{bs}{8-s}$. If $s = 3$, by Theorem 3.1, the only case that applies is given by eqn. (2b), and in this case $r \le \frac{3b}{5}$. Note that the three sets are pairwise disjoint since $b - 4r_1 = 2b - 4r_2$ or $3b - 4r_3 = 2b - 4r_2$ are impossible because b is odd and $b - 4r_1 = 3b - 4r_3$ is impossible because $4 \nmid 2b$. □

Recall that $\mathrm{F}(S) \in \mathrm{ORD}(S)$, so that $\mathrm{F}(S)$ is the largest element in $\mathrm{ORD}(S)$, which is a subset of $\mathrm{G}(S)$. We now turn our attention to determining the smallest element in $\mathrm{ORD}(S)$, in the special case $S = \langle a, b\rangle$. From Corollaries 3.3, 3.4 and 3.5, we have

$$\begin{aligned}
\min \mathrm{ORD}(\langle 2, b\rangle) &= b - 2\lfloor \tfrac{b}{3} \rfloor, && \text{(3a)}\\
\min \mathrm{ORD}(\langle 3, b\rangle) &= \min\left\{b - 3\lfloor \tfrac{b}{5} \rfloor, 2b - 3\lfloor \tfrac{b}{2} \rfloor\right\} \\
&= b - 3\lfloor \tfrac{b}{5} \rfloor \ \text{ except when } \ b \in \{4, 8, 14\}, && \text{(3b)}\\
\min \mathrm{ORD}(\langle 4, b\rangle) &= \min\left\{b - 4\lfloor \tfrac{b}{7} \rfloor, 2b - 4\lfloor \tfrac{b}{3} \rfloor, 3b - 4\lfloor \tfrac{3b}{5} \rfloor\right\} \\
&= b - 4\lfloor \tfrac{b}{7} \rfloor \ \text{ except when } \ b = 5. && \text{(3c)}
\end{aligned}$$

Henceforth, we take $S = \langle a, b\rangle$, and assume $3 < a < b$.

Theorem 3.6 *Let a, b be positive integers with $3 < a < b$ and $\gcd(a, b) = 1$. If $\min ORD(\langle a, b\rangle) = m$, then*

$$m = \begin{cases}
a - 1 & \text{if } b = a + 1; && \text{(4a)}\\
2b - a & \text{if } a + 2 \le b \le a + \lfloor \tfrac{a}{3} \rfloor; && \text{(4b)}\\
3a - b & \text{if } a + \lfloor \tfrac{a}{3} \rfloor + 1 \le b \le a + \lfloor \tfrac{a+1}{2} \rfloor + 1; && \text{(4c)}\\
2b - a & \text{if } a + \lfloor \tfrac{a+1}{2} \rfloor + 2 \le b \le a + \lfloor \tfrac{2a}{3} \rfloor; && \text{(4d)}\\
4a - b & \text{if } a + \lfloor \tfrac{2a}{3} \rfloor + 1 \le b \le 2a - 3; && \text{(4e)}\\
b - 2 & \text{if } b = 2a - 2; && \text{(4f)}\\
b - \lfloor \tfrac{b}{2a-1} \rfloor a & \text{if } b \ge 2a - 1. && \text{(4g)}
\end{cases}$$

The only four instances of (a, b) that appear in more than one case, together with the formula that applies, are as follows:
eqn. (4c) *applies when* $(a, b) = (7, 12)$,
eqn. (4f) *applies when* $(a, b) = (5, 8)$,
eqn. (4g) *applies when* $(a, b) = (4, 7)$ *or* $(5, 9)$.

The intervals given by the cases in eqns. (4a)–(4g) would have been pairwise disjoint had each interval been non-empty. Assuming $\gcd(a, b) = 1$, the condition required for the intervals given by eqn. (4b), eqn. (4d), eqn. (4e) to be non-empty are $a \ge 7$, $a \ge 14$, $a \ge 10$, respectively. So the intervals are certainly pairwise disjoint if $a \ge 14$.

Table 1 Equivalent conditions for $f(s) < 0$

$f(1) < 0$	$(2a-1)d < (a-1)b$
$f(a-1) < 0$	$(a+1)d < (a-1)b$
$f(2) < 0$	$(a-1)d < (a-2)b$
$f(3) < 0$	$(2a-3)d < 3(a-3)b$
$f(a-2) < 0$	$(a+2)d < 2(a-2)b$
$f(a-3) < 0$	$(a+3)d < 3(a-3)b$

If $10 \le a \le 13$, the interval given by eqn. (4d) is empty, and $a + \lfloor \frac{a+1}{2} \rfloor + 1 < a + \lfloor \frac{2a}{3} \rfloor + 1$ in this interval. So there is no value of a that appears in more than one interval in this range.

If $7 \le a \le 9$, the intervals given by eqn. (4d) and eqn. (4e) are empty, and $a + \lfloor \frac{a+1}{2} \rfloor + 1 < 2a - 2$ for $a = 8, 9$. However, $a = 7$ is a part of the intervals given by eqn. (4c) and eqn. (4f), with $b = 12$ correspondingly.

If $4 \le a \le 6$, the intervals given by eqn. (4b), eqn. (4d) and eqn. (4e) are empty, and $a + 1 < a + \lfloor \frac{a+1}{2} \rfloor + 1$ for any value of a. However, the intervals given by eqn. (4c) and eqn. (4f) intersect at $a = 4$ (with $b = 7$) and at $a = 5$ (with $b = 8, 9$), but not at $a = 6$ because the only corresponding b is not permissible.

These account for the four pairs (a, b) that appear in more than one interval.

Let $S = \langle a, b \rangle$. If $d \in \mathrm{ORD}(S)$, then $d \notin S$, and so we can write $d = bs - ar$ with $1 \le s \le a - 1$ and $1 \le r < bs/a$. We use Theorem 3.1 to show that the supposed minimum element is actually in $\mathrm{ORD}(S)$, and we use Proposition 3.2 to show that each smaller positive integer is not in $\mathrm{ORD}(S)$.

The organisation of the proof of Theorem 3.6 is as follows. The proof of $m \in \mathrm{ORD}(S)$ corresponding to cases given by the eqns. (4a)-(4g) is provided case by case, and appears in the second part. The proof for $d \notin \mathrm{ORD}(S)$ for $d < m$ appears in two parts, with the second part merged with the proof that $m \in \mathrm{ORD}(S)$. The first part of the proof for $d \notin \mathrm{ORD}(S)$ for $d < m$ follows a common theme and appears at the beginning of this proof.

We write $f(s) = bs^2 - (ab + d)s + 2ad$, $1 \le s \le a - 1$. In Table 1, we capture equivalent conditions for $f(s) < 0$. The row with $f(s_1) < 0$ appears before the row with $f(s_2) < 0$ if the upper bound for d/b in the earlier row is less than the upper bound for d/b in the latter row. Thus, for fixed values of b and d, $f(s_1) < 0$ implies $f(s_2) < 0$.

If $s = 1$, then $1 \le r \le b/(2a - 1)$ by eqn. (1a). Therefore $b - ar \notin \mathrm{ORD}(S)$ if $b < 2a - 1$. If $b \ge 2a - 1$, then $b - ar \notin \mathrm{ORD}(S)$ if and only if $b - ar < b - \lfloor \frac{b}{2a-1} \rfloor a$. So if $d = b - ar < m$, then $d \notin \mathrm{ORD}(S)$ in each of the cases in the Theorem.

If $s = 2$, then $1 \le r \le b/(a - 1)$ by eqns. (1a) and (1b). Therefore $2b - ar \notin \mathrm{ORD}(S)$ if and only if $2b - ar < 2b - \lfloor \frac{b}{a-1} \rfloor a$. Since $3a - b < 2b - a$ in the range given by eqn. (4c) and $4a - b < 2b - a$ in the range given by eqn. (4e), and $a - 1 < 2b - a$ for $b = a + 1$, we have $m \le 2b - a$ if $a + 1 \le b < 2(a - 1)$. So if $d < m$ and $a + 1 \le b < 2(a - 1)$, then $d < 2b - a = 2b - \lfloor \frac{b}{a-1} \rfloor a$, so that $d \notin \mathrm{ORD}(S)$. If $b = 2a - 2$ and $2b - ar = d < m = b - 2$, then $r > 2$. Hence $d \notin \mathrm{ORD}(S)$ if $d < m$ and $b = 2a - 2$. Suppose $b \ge 2a - 1$ and $2b - ar < m = b - \lfloor \frac{b}{2a-1} \rfloor a$. Then

$r \geq \lfloor \frac{b}{a} \rfloor + \lfloor \frac{b}{2a-1} \rfloor + 1$; it suffices to prove that this is at least $\lfloor \frac{b}{a-1} \rfloor + 1$. Define

$$D(b) = \lfloor \tfrac{b}{a} \rfloor + \lfloor \tfrac{b}{2a-1} \rfloor - \lfloor \tfrac{b}{a-1} \rfloor,$$

with $b \geq 2a-1$. Note that $0 \leq \lfloor \frac{b}{a-1} \rfloor - \lfloor \frac{b}{a} \rfloor \leq 1 \leq \lfloor \frac{b}{2a-1} \rfloor$ if $2a-1 \leq b \leq a(a-1)+1$ and $0 \leq \lfloor \frac{b}{a-1} \rfloor - \lfloor \frac{b}{a} \rfloor \leq 2 \leq \lfloor \frac{b}{2a-1} \rfloor$ if $a(a-1)+2 \leq b < a(a-1)+2a-1$. Hence, $D(b) \geq 0$ for $2a-1 \leq b < a(a-1)+2a-1$. Moreover, $D\big(b+a(a-1)\big) - D(b) = \Big(\lfloor \frac{b+a(a-1)}{2a-1} \rfloor - \lfloor \frac{b}{2a-1} \rfloor \Big) - 1 \geq 0$ since $a(a-1) \geq 2a-1$ for $a \geq 3$. This proves $D(b) \geq 0$ for $b \geq 2a-1$. Hence $d \notin \mathrm{ORD}(S)$ if $d < m$ and $b \geq 2a-1$. So if $d = 2b - ar < m$, then $d \notin \mathrm{ORD}(S)$ in each of the cases in the Theorem.

If $s = a-1$, then $1 \leq r \leq (a-1)b/(a+1)$ by eqns. (2a) and (2b). Therefore $(a-1)b - ar \notin \mathrm{ORD}(S)$ if and only if $(a-1)b - ar < (a-1)b - \lfloor \frac{(a-1)b}{a+1} \rfloor a$.

For the cases given by eqn. (4b) and eqn. (4d), $b(a-1) - ar = d < m = 2b - a$ gives $r > b + 1 - \frac{3b}{a}$. For $b \in \big[a+2, a + \lfloor \frac{a}{3} \rfloor\big]$, we have $3 + \frac{6}{a} \leq \frac{3b}{a} \leq 4$, and so $r \geq b - 2 > b - \frac{2b}{a+1} = \frac{(a-1)b}{a+1}$.

For $b \in \big[a + \lfloor \frac{a+1}{2} \rfloor + 2, a + \lfloor \frac{2a}{3} \rfloor\big]$, we have $\frac{9}{2} + \frac{6}{a} \leq \frac{3b}{a} \leq 5$, and so $r \geq b-3 > b - \frac{2b}{a+1}$ since $3(a+1) < 2b$ in this case.

For the case given by eqn. (4c), $b(a-1) - ar = d < m = 3a - b$ gives $r \geq b - 2 > b - \frac{2b}{a+1} = \frac{(a-1)b}{a+1}$.

For the case given by eqn. (4e), $b(a-1) - ar = d < m = 4a - b$ gives $r \geq b - 3 > b - \frac{2b}{a+1}$ since $3(a+1) < 2b$ in this case.

For the case given by eqn. (4g), $b(a-1) - ar = d < m = b - \lfloor \frac{b}{2a-1} \rfloor a$ gives $r > b + \lfloor \frac{b}{2a-1} \rfloor - \frac{2b}{a}$. Now $b + \lfloor \frac{b}{2a-1} \rfloor - \frac{2b}{a} \geq b - \frac{2b}{a+1}$ if and only if $\lfloor \frac{b}{2a-1} \rfloor \geq \frac{2b}{a(a+1)}$. To see when the equivalent inequality holds, first consider $b = k(2a-1)$ with $k \geq 1$. Then the inequality reduces to $ka(a+1) \geq 2k(2a-1)$, and hence to $(a-1)(a-2) \geq 0$, which is true. Otherwise $b = (k+1)(2a-1) - x$, with $1 \leq x < 2a-1$, and the inequality reduces to $ka(a+1) \geq 2(k+1)(2a-1) - 2x$, and hence to $k(a-1)(a-2) + 2x \geq 2(2a-1)$. Now $k(a-1)(a-2) + 2x - 2(2a-1) \geq (a-1)(a-2) + 2 - 2(2a-1) = (a-1)(a-6)$ which is greater than or equal to 0 for $a \geq 6$. Thus, it remains to consider the cases where $a \in \{4, 5\}$. In addition, if $k > 1$, then $k(a-1)(a-2) + 2x - 2(2a-1) \geq 2(a-1)(a-2) + 2 - 2(2a-1) = (a-1)(a-4) \geq 0$ for $a \geq 4$. Hence we may only consider the values of b (coprime to a) corresponding to $k = 1$. When $k = 1$, the inequality to verify further reduces to $(a-1)(a-2) + 2x \geq 2(2a-1)$, or to $a^2 - 7a + 4 + 2x \geq 0$. If $a = 4$, this holds for $x \geq 4$; if $a = 5$, this holds for $x \geq 3$. Therefore, the only remaining cases that do not satisfy the inequality are $a = 4$, $b = 11, 13$, and $a = 5$, $b = 16, 17$. For these four cases, we may verify individually that whenever $r > b + \lfloor \frac{b}{2a-1} \rfloor - \frac{2b}{a} = b + 1 - \frac{2b}{a}$, then $r > b - \frac{2b}{a+1}$ since r is an integer.

So if $d = (a-1)b - ar < m$, then $d \notin \mathrm{ORD}(S)$ in each of the cases in the Theorem.

In the second part of the proof that follows, we show that $m \in \mathrm{ORD}(S)$ and also that $f(3) < 0$ in all cases except those given by eqn. (4a) and eqn. (4f); in these exceptional cases, $d \notin \mathrm{ORD}(S)$ for $d < m$ is directly established. Since we have shown that $f(s) < 0$ for $s \in \{1, 2, a-1\}$ in the first part of the proof in all cases

except those given by eqn. (4a) and eqn. (4f), Table 1 would show that $f(s) < 0$ for $s \in \{1, \ldots, a-1\}$.

***Proof of eqn.* (4a)** Let $b = a+1$. We show that $a-1 \in \mathrm{ORD}(S)$ and that $d \notin \mathrm{ORD}(S)$ for $1 \le d \le a-2$. If $d \le a-1$, then $s = r = d$.

If $d = a-1$, part (II) applies and $\mu = \frac{a-1}{2}$. If a is odd, we need to verify $r(\lfloor\mu\rfloor + 1) \le \lfloor\mu\rfloor b$ by eqn. (2a). This amounts to verifying $(a-1)\frac{a+1}{2} \le \frac{a-1}{2}(a+1)$. If a is even, we need to verify $r(\lfloor 2\mu\rfloor + 2) \le \lfloor 2\mu\rfloor b$ by eqn. (2b). This amounts to verifying $(a-1)(a+1) \le (a-1)(a+1)$. Both inequalities hold, so we have $a-1 \in \mathrm{ORD}(S)$.

To show that $d \notin \mathrm{ORD}(S)$ for $1 \le d \le a-2$, we apply Proposition 3.2. We need to verify $(a-d)d > d$, or that $a - d > 1$, which is true. This proves the result in eqn. (4a).

***Proof of eqn.* (4b) & *eqn.* (4d)** Let $a+2 \le b \le a+\lfloor\frac{a}{3}\rfloor$ or $a+\lfloor\frac{a+1}{2}\rfloor+2 \le b \le a+\lfloor\frac{2a}{3}\rfloor$. The first inequality holds only for $a \ge 7$ (if $a = 6$, then $b = 8$), and the second inequality holds only for $a \ge 14$. We show that $2b-a \in \mathrm{ORD}(S)$ and that $d \notin \mathrm{ORD}(S)$ for $1 \le d \le 2b-a-1$.

If $d = 2b-a$, then $s = 2$ and $r = 1$, and so part (I) applies since we may assume $a > 3$. If a is even, we need to verify $r\left(2\frac{a}{s} - 1\right) \le b$ by eqn. (1a). If a is odd, we need to verify $2r\lfloor\frac{a}{s}\rfloor \le b$ by eqn. (1b). In both cases, we need to show $a - 1 \le b$. Since this holds, we have $2b - a \in \mathrm{ORD}(S)$.

Now suppose $1 \le d = bs - ar < 2b - a = m$. If $s = 1$, then $d \notin \mathrm{ORD}(S)$ when $r > b/(2a-1)$ by Theorem 3.1, which is the same as $d \notin \mathrm{ORD}(S)$ when $d < m$ for the case $s = 1$.

We now claim that $f(3) < 0$, or that $(2a-3)d < 3(a-3)b$, by Table 1. For this, it is enough to show that $(2a-3)(m-1) < 3(a-3)b$, or that $(a+3)b < (a+1)(2a-3)$ for $a+2 \le b \le a+\lfloor\frac{a}{3}\rfloor$ and for $a+\lfloor\frac{a+1}{2}\rfloor+2 \le b \le a+\lfloor\frac{2a}{3}\rfloor$.

The first of these is equivalent to showing the above inequality holds for $b = a+\lfloor\frac{a}{3}\rfloor$. Replacing $\lfloor\frac{a}{3}\rfloor$ by $\frac{a}{3}$ and simplifying shows this inequality to hold precisely when $2a^2 - 15a - 9 > 0$. This is true for $a > 8$, and so the claim holds for $a > 8$. We list the cases corresponding to $a \in \{7, 8\}$.

The second of these is equivalent to showing the above inequality holds for $b = a+\lfloor\frac{2a}{3}\rfloor$. Replacing $\lfloor\frac{2a}{3}\rfloor$ by $\frac{2a}{3}$ and simplifying shows this inequality to hold precisely when $a^2 - 18a - 9 > 0$. This is true for $a > 18$, and so the claim holds for $a > 18$. We list the cases corresponding to $a \in \{14, 15, 16, 17, 18\}$ (Table 2).

This proves the results in eqns. (4b) and (4d).

***Proof of eqn.* (4c)** Let $a+\lfloor\frac{a}{3}\rfloor+1 \le b \le a+\lfloor\frac{a+1}{2}\rfloor+1$. We show that $3a-b \in \mathrm{ORD}(S)$ and that $d \notin \mathrm{ORD}(S)$ for $1 \le d \le 3a-b-1$.

If $d = 3a-b$, then $s = a-1$ and $r = b-3$; we may assume $b > 3$ since we have $a > 3$. Now part (II) applies and $\mu = \frac{a-1}{2}$. If a is odd, we need to verify $r(\lfloor\mu\rfloor + 1) \le \lfloor\mu\rfloor b$ by eqn. (2a). This amounts to verifying $(b-3)\frac{a+1}{2} \le \frac{a-1}{2}b$. If a is even, we need to verify $r(\lfloor 2\mu\rfloor + 2) \le \lfloor 2\mu\rfloor b$ by eqn. (2b). This amounts to verifying $(b-3)(a+1) \le (a-1)b$. Both inequalities hold precisely when $b \le \frac{3}{2}(a+1)$, and so we have $3a - b \in \mathrm{ORD}(S)$.

Table 2 Exceptional cases for eqn. (4d)

a	b	$(a+3)b < (a+1)(2a-3)$	$\frac{3(a-3)b}{2a-3} \le d \le m-1$ $d \notin \langle a, b\rangle$	$\mathrm{G}\left(\frac{1}{d}\langle a, b\rangle\right)$
7	9	fails	10	$\{1, 2, 4\}$
8	DNE	x	x	x
14	23	fails	31	$\{1, 2, 4, 5, 8\}$
15	DNE	x	x	x
16	DNE	x	x	x
17	28	fails	38	$\{1, 2, 3, 4, 5, 6, 10\}$
18	DNE	x	x	x

Table 3 Exceptional cases for eqn. (4c)

a	b	$(2a-3)(3a-1) < (5a-12)b$	$\frac{3(a-3)b}{2a-3} \le d \le m-1$ $d \notin \langle a, b\rangle$	$\mathrm{G}\left(\frac{1}{d}\langle a, b\rangle\right)$
4	7	holds	x	x
5	7	fails	6	$\{1, 3\}$
	8	holds	x	x
	9	holds	x	x
6	DNE	x	x	x

Now suppose $1 \le d = bs - ar < 3a - b = m$. If $s = 1$, then $d \notin \mathrm{ORD}(S)$ when $r > b/(2a-1)$ by Theorem 3.1, which is the same as $d \notin \mathrm{ORD}(S)$ when $d < m$ for the case $s = 1$ (Table 3).

We now claim that $f(3) < 0$, or that $(2a-3)d < 3(a-3)b$, by Table 1. For this, it is enough to show that $(2a-3)(m-1) < 3(a-3)b$, or that $(2a-3)(3a-1) < (5a-12)b$ for $a + \lfloor \frac{a}{3} \rfloor + 1 \le b \le a + \lfloor \frac{a+1}{2} \rfloor + 1$. This is equivalent to showing the above inequality holds for $b = a + \lfloor \frac{a}{3} \rfloor + 1$. Replacing $\lfloor \frac{a}{3} \rfloor$ by $\frac{a-2}{3}$ and simplifying shows this inequality to hold precisely when $2a^2 - 10a - 21 > 0$. This is true for $a > 6$, and so the claim holds for $a > 6$. We note that the cases $(a, b) \in \{(4, 7), (5, 8), (5, 9)\}$ are covered by eqn. (4f) or eqn. (4g).

This proves the result in eqn. (4c).

***Proof of eqn.* (4e)** Let $a + \lfloor \frac{2a}{3} \rfloor + 1 \le b \le 2a - 3$. The inequality holds only for $a \ge 10$. We show that $4a - b \in \mathrm{ORD}(S)$ and that $d \notin \mathrm{ORD}(S)$ for $1 \le d \le 4a - b - 1$.

If $d = 4a - b$, then $s = a - 1$ and $r = b - 4$; we may assume $b > 4$ since we have $a > 3$. Now part (II) applies and $\mu = \frac{a-1}{2}$. If a is odd, we need to verify $r(\lfloor \mu \rfloor + 1) \le \lfloor \mu \rfloor b$ by eqn. (2a). This amounts to verifying $(b-4)\frac{a+1}{2} \le \frac{a-1}{2}b$. If a is even, we need to verify $r(\lfloor 2\mu \rfloor + 2) \le \lfloor 2\mu \rfloor b$ by eqn. (2b). This amounts to verifying $(b-4)(a+1) \le (a-1)b$. Both inequalities hold precisely when $b \le 2(a+1)$, and so we have $4a - b \in \mathrm{ORD}(S)$.

Table 4 Exceptional cases for eqn. (4e)

a	b	$(2a-3)(4a-1) < (5a-12)b$	$\frac{3(a-3)b}{2a-3} \le d \le m-1$ $d \notin \langle a, b\rangle$	$\mathrm{G}\left(\frac{1}{d}\langle a, b\rangle\right)$
10	17	fails	22	$\{1, 3\}$
11	19	fails	24	$\{1, 2, 3, 7\}$
12	DNE	x	x	x
13	22	fails	29	$\{1, 2, 4, 7\}$
	23	holds	x	x
14	25	holds	x	x

Now suppose $1 \le d = bs - ar < 4a - b = m$. If $s = 1$, then $d \notin \mathrm{ORD}(S)$ when $r > b/(2a-1)$ by Theorem 3.1, which is the same as $d \notin \mathrm{ORD}(S)$ when $d < m$ for the case $s = 1$ (Table 4).

We now claim that $f(3) < 0$, or that $(2a-3)d < 3(a-3)b$, by Table 1. For this, it is enough to show that $(2a-3)(m-1) < 3(a-3)b$, or that $(2a-3)(4a-1) < (5a-12)b$ for $a + \lfloor \frac{2a}{3} \rfloor + 1 \le b \le 2a - 3$. This is equivalent to showing the above inequality holds for $b = a + \lfloor \frac{2a}{3} \rfloor + 1$. Replacing $\lfloor \frac{2a}{3} \rfloor$ by $\frac{2a-2}{3}$ and simplifying shows this inequality to hold precisely when $a^2 - 13a - 21 > 0$. This is true for $a > 14$, and so the claim holds for $a > 14$.

This proves the result in eqn. (4e).

***Proof of eqn.* (4f)** Let $b = 2a-2$, so that a must be odd. We show that $b-2 \in \mathrm{ORD}(S)$ and that $d \notin \mathrm{ORD}(S)$ for $1 \le d \le b-3$. If $a = 3$, then $b = a + 1$, which is covered by part (i) (since $a - 1 = b - 2$ in this case). Henceforth, we may assume $a > 3$.

If $d = b-2$, then $r = s = 2$, and so part (I) applies. Since $a \pmod s = 1$, we need to verify $r\left(2\lfloor \frac{a}{s} \rfloor\right) \le b$ by eqn. (1b), or that $2 \cdot 2^{\frac{a-1}{2}} \le 2a - 2$. Since this inequality holds, we have $b - 2 \in \mathrm{ORD}(S)$.

Henceforth suppose $1 \le d \le b-3$. Observe that

$$1 = b\left(\tfrac{a-1}{2}\right) - a(a-2). \tag{5}$$

Two cases arise: (i) d is even, (ii) d is odd.

Case (i). From eqn. (5), $s \equiv \frac{a-1}{2}d \equiv -\frac{d}{2} \pmod a$, so that $s = a - \frac{d}{2}$ since $d < 2a$. Thus

$$r = \tfrac{bs-d}{a} = \tfrac{1}{a}\left(2(a-1)\left(a - \tfrac{d}{2}\right) - d\right) = 2(a-1) - d = b - d.$$

By Proposition 3.2, we need to verify $(a-s)r > d$, or $r > 2$, which is true.

Case (ii). From eqn. (5), $s \equiv \frac{a-1}{2}d \equiv \frac{a-1}{2}(d-1) + \frac{a-1}{2} \equiv \frac{a-1}{2} - \frac{d-1}{2} \equiv \frac{a-d}{2} \pmod a$. If $d < a$, then $s = \frac{a-d}{2}$. Thus

$$r = \tfrac{bs-d}{a} = \tfrac{1}{a}\left(2(a-1)\cdot\tfrac{a-d}{2} - d\right) = a - 1 - d.$$

By Proposition 3.2, we need to verify $(a-s)r > d$, or $(a+d)(a-d-1) > 2d$. This is true since $d < a$ implies $d \le a - 2$ as a, d are both odd.

If $d > a$, then $s = a - \frac{d-a}{2} = \frac{3a-d}{2}$. Thus

$$r = \frac{bs-d}{a} = \frac{1}{a}\left(2(a-1)\cdot\frac{3a-d}{2} - d\right) = 3(a-1) - d.$$

By Proposition 3.2, we need to verify $(a-s)r > d$, or $(d-a)(3a-d-3) > 2d$. Since $(d-a)+(3a-d-3) = 2a-3$ and $d-a \geq 2$, we have $(d-a)(3a-d-3) \geq 2(2a-5)$, with equality if and only if $d-a = 2$. Since $d \leq b-3 = 2a-5$ and d is odd, the only exception to the strict inequality above is when $d = a+2$ and $d = 2a-5$, or when $a = 7$ and $d = 9$. Thus, except when $(a, b, d) = (7, 12, 9)$, $d \notin \mathrm{ORD}(S)$ for $1 \leq d \leq b-3$. This proves the result in eqn. (4f), with the exception of the triple noted above.

***Proof of eqn.* (4g)** Let $b \geq 2a-1$. We show that $b - \lfloor \frac{b}{2a-1} \rfloor a \in \mathrm{ORD}(S)$ and that $d \notin \mathrm{ORD}(S)$ for $1 \leq d < b - \lfloor \frac{b}{2a-1} \rfloor a$. Since the result holds for $a = 4$ by eqn. (3c), we assume that $a > 4$.

If $d = b - \lfloor \frac{b}{2a-1} \rfloor a$, then $s = 1$ and $r = \lfloor \frac{b}{2a-1} \rfloor$, and so part (I) applies. We need to verify $r\left(2\frac{a}{s} - 1\right) \leq b$ by eqn. (1a), or that $r \leq b/(2a-1)$. Since this holds, we have $b - \lfloor \frac{b}{2a-1} \rfloor a \in \mathrm{ORD}(S)$.

Now suppose $1 \leq d = bs - ar < b - \lfloor \frac{b}{2a-1} \rfloor a = m$. If $s = 1$, then $d \notin \mathrm{ORD}(S)$ when $r > b/(2a-1)$ by Theorem 3.1, which is the same as $d \notin \mathrm{ORD}(S)$ when $d < m$ for the case $s = 1$.

We now claim that $f(a-1) < 0$, or that $(a+1)d < (a-1)b$, by Table 1. For this, it is enough to show that $(a+1)(m-1) < (a-1)b$ for each $b \geq 2a-1$. Fix $a > 3$, and write

$$D(b) = (a-1)b - (a+1)(m-1).$$

If $2a-1 \leq b < 2(2a-1)$, then $D(b) = (a-1)b - (a+1)(b-a-1) = (a+1)^2 - 2b \geq (a+1)^2 - 2(4a-3) = (a-3)^2 - 2 > 0$ except when $a = 4$. Moreover, $D(b+2a-1) - D(b) = (a-1)(2a-1) - (a+1)(a-1) = (a-1)(a-2) > 0$ for each $a > 3$. Therefore, $D(b) > 0$ for $b \geq 2a-1$ if $a > 4$.

This proves the result in eqn. (4g).

4 Estimation of $\left|\mathrm{ORD}(\langle a, b\rangle)\right|$

Theorem 3.1 may be used to determine $\mathrm{ORD}(\langle a, b\rangle)$. For a specific value of a, and for each $s \in \{1, \ldots, a-1\}$, $s \neq \frac{a}{2}$, exactly one of eqns. (1a)–(1c), eqns. (2a)–(2b) apply. When a is even and $s = \frac{a}{2}$, eqn. (1a) and exactly one of eqns. (2a)–(2b) applies. In each case, these give an upper bound for r, as shown in Corollaries 3.3 & 3.4. On the other hand, for general a, b, Theorem 3.1 does not yield an explicit description of $\mathrm{ORD}(\langle a, b\rangle)$. We turn our attention to estimating $\left|\mathrm{ORD}(\langle a, b\rangle)\right|$. Since $\mathrm{ORD}(\langle a, b\rangle) \subseteq \mathrm{G}(\langle a, b\rangle)$ and $\left|\mathrm{G}(\langle a, b\rangle)\right| = \mathrm{g}(\langle a, b\rangle) = (a-1)(b-1)/2$, we readily have the upper bound $ab/2$. From Corollaries 3.3, 3.4 and 3.5, we have

$$\left|\mathrm{ORD}(\langle 2, b\rangle)\right| = \left\lfloor \tfrac{b}{3} \right\rfloor, \tag{6a}$$

$$\left|\mathrm{ORD}(\langle 3, b\rangle)\right| = \left\lfloor \tfrac{b}{2} \right\rfloor + \left\lfloor \tfrac{b}{5} \right\rfloor, \tag{6b}$$

$$\left|\mathrm{ORD}(\langle 4, b\rangle)\right| = \left\lfloor \tfrac{b}{3} \right\rfloor + \left\lfloor \tfrac{b}{7} \right\rfloor + \left\lfloor \tfrac{3b}{5} \right\rfloor. \tag{6c}$$

Thus,

$$\tfrac{b-2}{3} \le \left|\mathrm{ORD}(\langle 2, b\rangle)\right| \le \tfrac{b}{3}, \quad \tfrac{7b-13}{10} \le \left|\mathrm{ORD}(\langle 3, b\rangle)\right| \le \tfrac{7b}{10},$$
$$\text{and } \tfrac{113b-244}{105} \le \left|\mathrm{ORD}(\langle 4, b\rangle)\right| \le \tfrac{113b}{105}.$$

More generally, we show that

$$c_1 ab \le \left|\mathrm{ORD}(\langle a, b\rangle)\right| \le c_2 ab,$$

where c_1 may be chosen arbitrarily close to, but smaller than, $\frac{25}{16} - \log 4 \approx 0.1762$ and applying to all pairs a, b with sufficiently large a, and $c_2 = \log 4 - 1 \approx 0.3863$, holding for all pairs a, b.

Theorem 4.1 *Let a, b be positive integers such that $\gcd(a, b) = 1$, $a < b$, and let $\epsilon > 0$. Then*

$$\left(\tfrac{25}{16} - \log 4 - \epsilon\right) ab \le \left|ORD(\langle a, b\rangle)\right| \le (\log 4 - 1)ab,$$

where the upper bound holds for every pair a, b while the lower bound holds for all pairs a, b with $a \ge A(\epsilon)$.

Proof From Theorem 3.1, the number of elements in $\mathrm{ORD}(\langle a, b\rangle)$ equals the sum of the greatest integer less than or equal to the upper bounds for r in the cases given by eqns. (1a)–(2b).

For eqn. (1a), the upper bound is $bs/(2a - s)$. For eqn. (1b), the upper bound is obtained by replacing the denominator by $2\left(\frac{a}{s} - \frac{1}{2}\right) = \frac{2a-s}{s}$. Hence, the upper bound in this case is $bs/(2a - s)$. For eqn. (1c), the upper bound is obtained by replacing the denominator by $2\left(\frac{a}{s} - 1\right) + 1 = \frac{2a-s}{s}$. Hence, the upper bound in this case is $bs/(2a - s)$.

For eqns. (2a) & (2b), since the function $f(x) = x/(x + 1)$ is increasing in $(0, \infty)$, replacing $\lfloor \mu \rfloor$ by μ gives the upper bound $\frac{\mu}{\mu+1} b = \frac{s}{2a-s} b$.

Therefore, the upper bound in each case is $bs/(2a - s)$. Recall that the harmonic series $\langle H_n \rangle$ satisfies

$$\log n + \gamma + \tfrac{1}{2n+1} \le H_n = \sum_{k=1}^{n} \tfrac{1}{k} \le \log n + \gamma + \tfrac{1}{2n},$$

where γ is the Euler–Mascheroni constant.

Hence,

$$\left|\mathrm{ORD}(\langle a, b\rangle)\right| \le b \sum_{s=1}^{a-1} \tfrac{s}{2a-s}$$

$$
\begin{aligned}
&\leq b\sum_{s=1}^{a-1}\left(2a\tfrac{1}{2a-s}-1\right)\\
&\leq b\big(2a\left(H_{2a-1}-H_a\right)-(a-1)\big)\\
&\leq b\left(2a\left(\log\tfrac{2a-1}{a}-\tfrac{2a-3}{2(2a-1)(2a+1)}\right)-(a-1)\right)\\
&\leq b\left(2a\left(\log 2+\log\left(1-\tfrac{1}{2a}\right)\right)-(a-1)\right)\\
&\leq b\left((\log 4-1)a-2a\left(\tfrac{1}{2a}\right)+1\right)\\
&\leq (\log 4-1)ab.
\end{aligned}
$$

For eqn. (1a), the lower bound is $\frac{bs}{2a-s}-1$. For eqn. (1b), the lower bound is obtained by replacing the denominator by $2\frac{a}{s}$, and then subtracting 1. Hence, the lower bound in this case is $\frac{bs}{2a}-1$. For eqn. (1c), the lower bound is obtained by replacing the denominator by $2\left(\frac{a}{s}-\frac{1}{2}\right)+1=\frac{2a}{s}$, and then subtracting 1. Hence, a common lower bound for these cases is $\frac{bs}{2a}-1$.

For eqns. (2a) & (2b), since the function $f(x)=x/(x+1)$ is increasing in $(0,\infty)$, replacing $\lfloor\mu\rfloor$ by $\mu-1$ gives the lower bound $\lfloor\frac{\mu-1}{\mu}b\rfloor=\lfloor\left(3-\frac{2a}{s}\right)b\rfloor$ for which $\left(3-\frac{2a}{s}\right)b-1$ is a lower bound.

We work with the case a even. The contribution to the lower bound from eqns. (1a)–(1c) is

$$
\sum_{s=1}^{a/2}\left(\tfrac{bs}{2a}-1\right)=\tfrac{(a+2)b}{16}-\tfrac{a}{2}=\tfrac{ab}{16}+\tfrac{b}{8}-\tfrac{a}{2}. \tag{7E}
$$

The contribution to the lower bound from eqns. (2a) & (2b) is

$$
\begin{aligned}
&\sum_{s=\frac{a}{2}+1}^{a-1}\left(\left(3-\tfrac{2a}{s}\right)b-1\right)\\
=\;&\tfrac{3(a-2)b}{2}-2ab\left(H_{a-1}-H_{a/2}\right)-\tfrac{a}{2}+1\\
\geq\;&\tfrac{3ab}{2}-3b-2ab\left(\log\tfrac{2(a-1)}{a}+\tfrac{1}{2a-2}-\tfrac{1}{a+1}\right)-\tfrac{a}{2}+1\\
\geq\;&\tfrac{3ab}{2}-3b-2ab\left(\left(\log 2+\log\left(1-\tfrac{1}{a}\right)\right)+\tfrac{1}{2a-2}-\tfrac{1}{a+1}\right)-\tfrac{a}{2}+1\\
\geq\;&\left(\tfrac{3}{2}-\log 4\right)ab-3b+2ab\left(\tfrac{1}{a}+\tfrac{1}{2a^2}-\tfrac{1}{2(a-1)}+\tfrac{1}{a+1}\right)-\tfrac{a}{2}+1\\
=\;&\left(\tfrac{3}{2}-\log 4\right)ab+\left(\tfrac{1}{a}-\tfrac{1}{a-1}-\tfrac{2}{a+1}\right)b-\tfrac{a}{2}+1.
\end{aligned} \tag{8E}
$$

Let $\epsilon>0$. From (7E) & (8E), we have

$$\begin{aligned}
&\left|\mathrm{ORD}(\langle a,b\rangle)\right| \\
&\geq \left(\tfrac{25}{16}-\log 4\right)ab+\left(\tfrac{1}{8}+\tfrac{1}{a}-\tfrac{1}{a-1}-\tfrac{2}{a+1}\right)b-a+1 \\
&= \left(\tfrac{25}{16}-\log 4-\epsilon\right)ab+\left(\tfrac{1}{8a}+\tfrac{1}{a^2}-\tfrac{1}{a(a-1)}-\tfrac{2}{a(a+1)}-\tfrac{a-1}{ab}+\epsilon\right)ab. \qquad (9E)
\end{aligned}$$

The second term on the RHS of estimate (9E) is clearly nonnegative for all sufficiently large a.

The lower bound estimates are for even a. For odd a, estimates corresponding to eqn. (7E)–(9E) are as follows:

$$\sum_{s=1}^{(a-1)/2}\left(\tfrac{bs}{2a}-1\right) = \tfrac{ab}{16}-\tfrac{b}{16a}-\tfrac{a-1}{2}. \qquad (7O)$$

$$\sum_{s=\frac{a+1}{2}}^{a-1}\left(\left(3-\tfrac{2a}{s}\right)b-1\right) = \left(\tfrac{3}{2}-\log 4\right)ab-\left(\tfrac{1}{a-1}+\tfrac{1}{2}\right)b-\tfrac{a-1}{2}. \qquad (8O)$$

$$\begin{aligned}
&\left|\mathrm{ORD}(\langle a,b\rangle)\right| \\
&\geq \left(\tfrac{25}{16}-\log 4-\epsilon\right)ab+\left(\epsilon-\tfrac{1}{16a^2}-\tfrac{a-1}{ab}-\tfrac{1}{2a}-\tfrac{1}{a(a-1)}\right)ab. \qquad (9O)
\end{aligned}$$

The second term on the RHS of estimate (9O) is clearly nonnegative for all sufficiently large a, as in the case of even a. □

Remark 4.2 As an illustration of Theorem 4.1, we show that

$$\left|\mathrm{ORD}(\langle a,b\rangle)\right| \geq \left(\tfrac{3}{2}-\log 4\right)ab$$

for all a, b with $15 \leq a < b$ when a is even and for all a, b with $24 \leq a < b$ when a is odd.

For even a, by estimate (9E), with $\epsilon = \frac{1}{16}$, we show that

$$\tfrac{ab}{16}+\tfrac{b}{8}-(a-1)+\left(\tfrac{1}{a}-\tfrac{1}{a-1}-\tfrac{2}{a+1}\right)b \geq 0 \qquad (10)$$

for all a, b with $a \geq 15$. This is equivalent to showing that the inequality in (10) holds in the case $b = a+1$, for each $a \geq A$, which reduces to verifying

$$a^2-13a-14-\tfrac{32}{a-1}+\tfrac{16}{a} \geq 0$$

for all $a \geq 15$.

For odd a, by estimate (9O), with $\epsilon = \frac{1}{16}$, we show that

$$\tfrac{ab}{16}-\tfrac{b}{16a}-(a-1)-\left(\tfrac{1}{2}+\tfrac{1}{a-1}\right)b \geq 0 \qquad (11)$$

for all a, b with $a \geq 24$. This is equivalent to showing that the inequality in (11) holds in the case $b = a + 1$, for each $a \geq A$, which reduces to verifying

$$a^2 - 23a - 9 - \tfrac{32}{a-1} - \tfrac{1}{a} \geq 0$$

for all $a \geq 24$.

Acknowledgements The authors gratefully acknowledge the contributions of the anonymous referee for pointing out some typos and suggesting several minor modifications to the initial draft of this manuscript.

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