ON A LINEAR DIOPHANTINE PROBLEM INVOLVING THE FIBONACCI AND LUCAS SEQUENCES

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Abstract

For a positive and relatively prime set A, let $\Gamma(A)$ denote the set of integers that are formed by taking nonnegative integer linear combinations of integers in A. Then there are finitely many positive integers that do not belong to $\Gamma(A)$. For A, let $\mathtt{g}(A)$ and $\mathtt{n}(A)$ denote the largest integer and the number of integers that do not belong to $\Gamma(A)$, respectively. We determine both $\mathtt{g}(A)$ and $\mathtt{n}(A)$ for two sets that arise naturally from the Fibonacci sequence and the Lucas sequence.

1. Introduction

For a finite set $A = \{a_1, \ldots, a_k\}$ of positive integers with gcd $A := \gcd(a_1, \ldots, a_k) = 1$, let $\Gamma(A) := \{a_1x_1 + \cdots + a_kx_k : x_i \geq 0\}$. It is well-known that $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ is finite. Although it was Sylvester [8] who first asked to determine

$$\mathtt{g}(A) := \max \Gamma^c(A),$$

and who showed that $\mathtt{g}(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$, it was Frobenius who was largely instrumental in giving this problem the early recognition and it is after him that the problem is also named. The monograph on the Frobenius problem

[5] gives an extensive survey. Related to the Frobenius problem is the problem of determining $\mathtt{n}(A) := |\Gamma^c(A)|$. As in the case of determining $\mathtt{g}(A)$, it was Sylvester who showed that $\mathtt{n}(a_1, a_2) = \frac{1}{2}(a_1 - 1)(a_2 - 1)$.

For each nonzero residue class $\mathbf{C}$ modulo a_1, let $\mathbf{m_C}$ denote the least integer in $\Gamma(A) \cap \mathbf{C}$. The functions $\mathtt{g}$ and $\mathtt{n}$ are easily determined from the values of $\mathbf{m_C}$ by Lemma 1. Brauer and Shockley [1] proved (i) and Selmer [7] proved (ii); a short proof of both results may be found in [9].

Lemma 1. ([1, 7]) *Let $a \in A$. For each nonzero residue class $\mathbf{C}$ modulo a, let $\mathbf{m_C}$ denote the least integer in $\Gamma(A) \cap \mathbf{C}$. Then*

(i) $\mathtt{g}(A) = \max_{\mathbf{C}} \mathbf{m_C} - a$, *the maximum taken over all nonzero classes $\mathbf{C}$ modulo a;*

(ii) $\mathtt{n}(A) = \dfrac{1}{a}\sum_{\mathbf{C}} \mathbf{m_C} - \frac{1}{2}(a-1)$, *the sum taken over all nonzero classes $\mathbf{C}$ modulo a.*

Exact values of $\mathtt{g}(A)$ and $\mathtt{n}(A)$ are difficult to determine in general when $|A| > 2$; refer to [5] for a list of cases when these have been determined. In the absence of exact results, research on the Frobenius problem has often been focused on sharpening bounds on $\mathtt{g}(A)$ and $\mathtt{n}(A)$, and on algorithmic aspects. Although the running time of these algorithms is superpolynomial, Kannan [3] gave a method that solves the Frobenius problem in polynomial time for a *fixed* number of variables using the concept of covering radius and Ramírez Alfonsín [4] showed that the problem is `NP-hard` under Turing reduction.

The Fibonacci and the Lucas sequences, $\{\mathcal{F}_n\}_{n\geq 1}$ and $\{\mathcal{L}_n\}_{n\geq 1}$, are among the most well-known and well-studied sequences in mathematics. Marín, Ramírez Alfonsín and Revuelta [6] studied the problem of determining the function $\mathtt{g}\big(\mathcal{F}_i, \mathcal{F}_j, \mathcal{F}_k\big)$ and $\mathtt{n}\big(\mathcal{F}_i, \mathcal{F}_j, \mathcal{F}_k\big)$, and found exact values in the case when $j = i + 2$. Matthews [2] studied the problem of finding $\mathtt{g}\big(\{a, a+b, a\mathcal{F}_{k-1} + b\mathcal{F}_k\}\big)$, where $a > \mathcal{F}_k$ and $\gcd(a, b) = 1$, and gave exact values in these cases.

In this article, we consider sets that arise from taking linear combinations from these sequences. More specifically, for any pair of positive and coprime integers a and b, we consider the sets

$$F = \big\{a, a{+}b, 2a{+}3b, \ldots, \mathcal{F}_{2k-1}a{+}\mathcal{F}_{2k}b\big\}, \quad L = \big\{a, a{+}3b, 4a{+}7b, \ldots, \mathcal{L}_{2k-1}a{+}\mathcal{L}_{2k}b\big\},$$

and give exact values to the functions $\mathtt{g}$ and $\mathtt{n}$ for these sets.

2. The Fibonacci Case

The Fibonacci numbers $\{\mathcal{F}_n\}_{n\geq 0}$ are given by the second-order recurrence

$$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} \text{ for } n \geq 2, \quad \mathcal{F}_1 = \mathcal{F}_2 = 1.$$

We extend the sequence to $\mathcal{F}_0 := \mathcal{F}_2 - \mathcal{F}_1 = 0$ and $\mathcal{F}_{-1} := \mathcal{F}_1 - \mathcal{F}_0 = 1$. Binet's formula is

$$\mathcal{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1}$$

for $n \geq 0$, where α, β are the roots of the equation $x^2 - x - 1 = 0$, with $\alpha > \beta$.

Let a, b be positive integers such that $\gcd(a, b) = 1$. Consider the set

$$\begin{aligned} F &= \{\mathcal{F}_{-1}a + \mathcal{F}_0 b, \mathcal{F}_1 a + \mathcal{F}_2 b, \mathcal{F}_3 a + \mathcal{F}_4 b, \ldots, \mathcal{F}_{2k-1}a + \mathcal{F}_{2k}b\} \\ &= \{a, a+b, 2a+3b, \ldots, \mathcal{F}_{2k-1}a + \mathcal{F}_{2k}b\}. \end{aligned}$$

Thus $\mathtt{g}(F)$ denotes the largest integer N such that the equation

$$\begin{aligned} &(\mathcal{F}_{-1}a + \mathcal{F}_0 b)x_0 + (\mathcal{F}_1 a + \mathcal{F}_2 b)x_1 + \cdots + (\mathcal{F}_{2k-1}a + \mathcal{F}_{2k}b)x_k \\ &\qquad = \left(\sum_{i=0}^{k} \mathcal{F}_{2i-1}x_i\right) a + \left(\sum_{i=0}^{k} \mathcal{F}_{2i}x_i\right) b = N \end{aligned} \tag{2}$$

has no solution in nonnegative integers $x_0, x_1, \ldots, x_k$, and $\mathtt{n}(F)$ the number of such N.

Proposition 1. *Let $\alpha = (1+\sqrt{5})/2$ denote the positive root of $x^2 - x - 1 = 0$. The sequence of Fibonacci numbers $\{\mathcal{F}_n\}_{n\geq 0}$ satisfies:*

(i) $\sum_{n=1}^{\infty} (\alpha \cdot \mathcal{F}_{2n-1} - \mathcal{F}_{2n}) = 1$.

(ii) $\mathcal{F}_{2n} < 3\mathcal{F}_{2n-2}$ *for* $n \geq 3$.

(iii) $2\mathcal{F}_{2n-2} + 2\mathcal{F}_{2n} > \mathcal{F}_{2n+2}$ *for* $n \geq 1$.

(iv) $2\mathcal{F}_{2n} + \mathcal{F}_{2n+2} + \cdots + \mathcal{F}_{2n+2k-4} + 2\mathcal{F}_{2n+2k-2} \geq \mathcal{F}_{2n+2k}$ *for* $n \geq 0$, $k \geq 1$.

(v) $\{\alpha \cdot \mathcal{F}_{2n-1} - \mathcal{F}_{2n}\}_{n\geq 1}$ *is decreasing.*

Proof. We use Binet's formula (1). Recall that $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ are the roots of $x^2 - x - 1 = 0$.

(i)

$$\begin{aligned} \sum_{n=1}^{\infty} (\alpha \cdot \mathcal{F}_{2n-1} - \mathcal{F}_{2n}) &= \sum_{n=1}^{\infty} \frac{\alpha(\alpha^{2n-1} - \beta^{2n-1}) - (\alpha^{2n} - \beta^{2n})}{\alpha - \beta} \\ &= -\sum_{n=1}^{\infty} \beta^{2n-1} = -\frac{\beta}{1-\beta^2} = 1. \end{aligned}$$

(ii) For $n \geq 3$,

$$\mathcal{F}_{2n} = \mathcal{F}_{2n-1} + \mathcal{F}_{2n-2} = 2\mathcal{F}_{2n-2} + \mathcal{F}_{2n-3} < 3\mathcal{F}_{2n-2}.$$

(iii) For $n \geq 1$,

$$\mathcal{F}_{2n+2} = \mathcal{F}_{2n+1} + \mathcal{F}_{2n} = 2\mathcal{F}_{2n} + \mathcal{F}_{2n-1} < 2\mathcal{F}_{2n} + 2\mathcal{F}_{2n-2}.$$

(iv)

$$\begin{aligned} 2\mathcal{F}_{2n} + \mathcal{F}_{2n+2} + \cdots + \mathcal{F}_{2n+2k-4} + 2\mathcal{F}_{2n+2k-2} &> \mathcal{F}_{2n+1} + \mathcal{F}_{2n+2} + \mathcal{F}_{2n+4} + \cdots \\ &\qquad + \mathcal{F}_{2n+2k-4} + 2\mathcal{F}_{2n+2k-2} \\ &= \mathcal{F}_{2n+3} + \mathcal{F}_{2n+4} + \mathcal{F}_{2n+6+} + \cdots \\ &\qquad + \mathcal{F}_{2n+2k-4} + 2\mathcal{F}_{2n+2k-2} \\ &= \vdots \\ &= \mathcal{F}_{2n+2k-3} + 2\mathcal{F}_{2n+2k-2} \\ &= \mathcal{F}_{2n+2k}. \end{aligned}$$

(v)

$$\begin{aligned} \big(\alpha \cdot \mathcal{F}_{2n-1} - \mathcal{F}_{2n}\big) - \big(\alpha \cdot \mathcal{F}_{2n+1} - \mathcal{F}_{2n+2}\big) &= \mathcal{F}_{2n+1} - \alpha \cdot \mathcal{F}_{2n} \\ &= \frac{(\alpha^{2n+1} - \beta^{2n+1}) - \alpha(\alpha^{2n} - \beta^{2n})}{\alpha - \beta} \\ &= \beta^{2n} > 0. \end{aligned}$$

□

Proposition 2. *Let a, b be positive integers, with $\gcd(a, b) = 1$. Let*

$$F = \big\{\mathcal{F}_{-1}a + \mathcal{F}_0 b, \mathcal{F}_1 a + \mathcal{F}_2 b, \mathcal{F}_3 a + \mathcal{F}_4 b, \ldots, \mathcal{F}_{2k-1}a + \mathcal{F}_{2k}b\big\}.$$

For each nonzero y, the least integer in $\Gamma(F)$ congruent to by modulo a is given by

$$\mathbf{m}_{by} = a\left\lceil \frac{\mathcal{F}_{2k-1}y}{\mathcal{F}_{2k}} \right\rceil + by.$$

Proof. For $1 \leq y \leq a-1$, let $\mathbf{m}_{by}$ denote the least integer in $\Gamma(F)$ congruent to by modulo a. Fix y. By (2), any integer N in $\Gamma(F)$ is of the form

$$\left(\sum_{i=0}^{k} \mathcal{F}_{2i-1}x_i\right) a + \left(\sum_{i=0}^{k} \mathcal{F}_{2i}x_i\right) b.$$

If $N \equiv by \pmod{a}$, then $\sum_{i=0}^{k} \mathcal{F}_{2i}x_i \equiv y \pmod{a}$ since $\gcd(a, b) = 1$. Let $\sum_{i=0}^{k} \mathcal{F}_{2i}x_i = y + ta$, $t \geq 0$. For a fixed $t \geq 0$, we wish to minimize

$$x = \sum_{i=0}^{k} \mathcal{F}_{2i-1}x_i \;\text{ subject to }\; \sum_{i=0}^{k} \mathcal{F}_{2i}x_i = y + ta.$$

We claim that this minimum is given by

$$x_{\min} = \left\lceil \frac{\mathcal{F}_{2k-1}(y+ta)}{\mathcal{F}_{2k}} \right\rceil. \tag{3}$$

Since $\mathbf{m}_{by} = \min\{ax_{\min} + b(y+ta) : t \geq 0\}$, the verification of this claim proves that $\mathbf{m}_{by} = a\left\lceil \frac{\mathcal{F}_{2k-1}y}{\mathcal{F}_{2k}} \right\rceil + by$.

Let $y' = y + ta$. We show that

$$\frac{\alpha \cdot \mathcal{F}_{2k-1} - \mathcal{F}_{2k}}{\mathcal{F}_{2k}} y' \leq \alpha \cdot x_{\min} - y' < \frac{\alpha \cdot \mathcal{F}_{2k-1} - \mathcal{F}_{2k}}{\mathcal{F}_{2k}} y' + \alpha. \tag{4}$$

For any $x = \sum_{i=0}^{k} \mathcal{F}_{2i-1} x_i$, we have

$$\alpha \cdot x - y' = \sum_{i=0}^{k} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big) x_i \geq \big(\alpha \cdot \mathcal{F}_{2k-1} - \mathcal{F}_{2k}\big) \sum_{i=0}^{k} x_i \geq \frac{\alpha \cdot \mathcal{F}_{2k-1} - \mathcal{F}_{2k}}{\mathcal{F}_{2k}} y' \tag{5}$$

since $y' = \sum_{i=0}^{k} \mathcal{F}_{2i} x_i \leq \mathcal{F}_{2k} \sum_{i=0}^{k} x_i$ and since $\{\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\}_{i \geq 1}$ is decreasing by Proposition 1, part (v). This proves the first inequality in (4).

We choose the sequence $\{x_i\}_{i \geq 0}$ greedily. In other words, we choose the largest i (say, i_0) such that $\mathcal{F}_{2i} \leq y'$, and then let $x_{i_0} = \lfloor y'/\mathcal{F}_{2i_0} \rfloor$. Let $x_i = 0$ for $i_0 < i \leq k$. If $y'' = y' - \mathcal{F}_{2i_0} x_{i_0} > 0$, we repeat the above procedure with y'' replacing y'. The procedure terminates because $\mathcal{F}_2 = 1$. Note that the term x_0 does not appear in the expression for y'; so we choose $x_0 = 0$. Because of Proposition 1, part (ii), we must have $x_i \in \{0, 1, 2\}$ for each $i < k$.

If $x_1 < 2$, then by Proposition 1, part (i),

$$\begin{aligned}
\sum_{i=0}^{k-1} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big) x_i &< 2 \sum_{i=2}^{\infty} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big) + \big(\alpha \cdot \mathcal{F}_1 - \mathcal{F}_2\big) \\
&= 2\big(1 - (\alpha - 1)\big) + (\alpha - 1) \\
&= 2 - (\alpha - 1) < \alpha. \qquad (6)
\end{aligned}$$

Now suppose $x_1 = 2$. We show that

$$\sum_{i=0}^{k-1} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big) x_i < 2(\alpha - 1) + \sum_{i=2}^{k-1} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big).$$

Since

$$(\alpha - 1) + \sum_{i=2}^{k-1} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big) = \sum_{i=1}^{k-1} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big) < \sum_{i=1}^{\infty} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big) = 1,$$

the inequality in (6) also holds in this case. Observe that by Proposition 1, part (iii), there does not exist i for which $x_i = x_{i+1} = 2$. If $x_i < 2$ for $1 < i < k$, the claimed inequality holds. Otherwise, let i, j be the two largest integers less than k for which $x_i = x_j = 2$. By Proposition 1, part (iv), $x_m = 0$ for some m between i and j. Define a sequence $\{x'_n\}_{n=0}^{k-1}$ by letting $x'_j = x'_m = 1$ and $x'_n = x_n$ for all other n. Then $\sum_{n=0}^{k-1} \big(\alpha \cdot \mathcal{F}_{2n-1} - \mathcal{F}_{2n}\big)x'_n > \sum_{n=0}^{k-1} \big(\alpha \cdot \mathcal{F}_{2n-1} - \mathcal{F}_{2n}\big)x_n$ by Proposition 1, part (v). Thus the sequence $\{x'_n\}_{n=0}^{k-1}$ has no 2's for $n > i$ and $\sum_{n=0}^{k-1} \big(\alpha \cdot \mathcal{F}_{2n-1} - \mathcal{F}_{2n}\big)x'_n > \sum_{n=0}^{k-1} \big(\alpha \cdot \mathcal{F}_{2n-1} - \mathcal{F}_{2n}\big)x_n$. We now repeat the above argument to the sequence $x_0, x_1, x_2, \ldots, x_i = 2$ since these integers have also been chosen greedily. This process terminates with a sequence with no 2's beyond x'_1. The sum corresponding to these terms of the sequence is therefore at most $2(\alpha - 1) + \sum_{i=2}^{k-1} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big)$. This shows that the inequality in (6) also holds when $x_1 = 2$.

From (6) we now have

$$\alpha \cdot x - y' = \sum_{i=0}^{k} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big)x_i \leq \sum_{i=0}^{k-1} \big(\alpha \cdot \mathcal{F}_{2i-1} - \mathcal{F}_{2i}\big)x_i + \frac{(\alpha \cdot \mathcal{F}_{2k-1} - \mathcal{F}_{2k})y'}{\mathcal{F}_{2k}}$$

$$< \alpha + \frac{(\alpha \cdot \mathcal{F}_{2k-1} - \mathcal{F}_{2k})y'}{\mathcal{F}_{2k}}.$$

Since (5) holds for every x and $\alpha \cdot x_{\min} - y' \leq \alpha \cdot x - y'$, we also have the second inequality in (4). It follows from (4) that

$$\frac{\mathcal{F}_{2k-1}y'}{\mathcal{F}_{2k}} \leq x_{\min} < \frac{\mathcal{F}_{2k-1}y'}{\mathcal{F}_{2k}} + 1. \tag{7}$$

This proves the proposition. □

Proposition 3. *Let a, b be positive integers, with $\gcd(a, b) = 1$. Let*

$$F = \big\{\mathcal{F}_{-1}a + \mathcal{F}_0 b, \mathcal{F}_1 a + \mathcal{F}_2 b, \mathcal{F}_3 a + \mathcal{F}_4 b, \ldots, \mathcal{F}_{2k-1}a + \mathcal{F}_{2k}b\big\}.$$

Then

(i)

$$\mathtt{g}(F) = a\left(\left\lceil \frac{\mathcal{F}_{2k-1}(a-1)}{\mathcal{F}_{2k}} \right\rceil - 1\right) + (a-1)b.$$

(ii)

$$\mathtt{n}(F) = \sum_{y=1}^{a-1} \left\lceil \frac{\mathcal{F}_{2k-1}y}{\mathcal{F}_{2k}} \right\rceil + \frac{1}{2}(a-1)(b-1).$$

Proof. This follows directly from Lemma 1 and Proposition 2.

(i)

$$\begin{aligned} \mathtt{g}\big(F\big) &= \max_{1\le y\le a-1} \mathbf{m}_{by} - a \\ &= \max_{1\le y\le a-1} \left(a \left\lceil \frac{\mathcal{F}_{2k-1}y}{\mathcal{F}_{2k}} \right\rceil + by \right) - a \\ &= a\left(\left\lceil \frac{\mathcal{F}_{2k-1}(a-1)}{\mathcal{F}_{2k}} \right\rceil - 1 \right) + (a-1)b. \end{aligned}$$

(ii)

$$\begin{aligned} \mathtt{n}\big(F\big) &= \frac{1}{a}\sum_{y=1}^{a-1} \mathbf{m}_{by} - \frac{1}{2}(a-1) \\ &= \frac{1}{a}\sum_{y=1}^{a-1} \left(a \left\lceil \frac{\mathcal{F}_{2k-1}y}{\mathcal{F}_{2k}} \right\rceil + by \right) - \frac{1}{2}(a-1) \\ &= \sum_{y=1}^{a-1} \left\lceil \frac{\mathcal{F}_{2k-1}y}{\mathcal{F}_{2k}} \right\rceil + \frac{b}{a}\sum_{y=1}^{a-1} y - \frac{1}{2}(a-1) \\ &= \sum_{y=1}^{a-1} \left\lceil \frac{\mathcal{F}_{2k-1}y}{\mathcal{F}_{2k}} \right\rceil + \frac{1}{2}(a-1)(b-1). \end{aligned}$$

□

Remark 1. *A simplification of the sum in Proposition 3, part (ii) appears difficult, but would be desirable.*

3. The Lucas Case

The `Lucas numbers` $\{\mathcal{L}_n\}_{n\ge 0}$ are given by the second-order recurrence

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} \ \text{ for } \ n \ge 2, \quad \mathcal{L}_1 = 1, \, \mathcal{L}_2 = 3.$$

Binet's formula is

$$\mathcal{L}_n = \alpha^n + \beta^n \tag{8}$$

for $n \ge 0$, where α, β are the roots of the equation $x^2 - x - 1 = 0$, with $\alpha > \beta$.

Let a, b be positive integers such that $\gcd(a, b) = 1$. Consider the set

$$\begin{aligned} L &= \big\{a, \mathcal{L}_1 a + \mathcal{L}_2 b, \mathcal{L}_3 a + \mathcal{L}_4 b, \ldots, \mathcal{L}_{2k-1} a + \mathcal{L}_{2k} b\big\} \\ &= \big\{a, a + 3b, 4a + 7b, \ldots, \mathcal{L}_{2k-1} a + \mathcal{L}_{2k} b\big\}. \end{aligned}$$

For the set involving Lucas sequences analogous to the Fibonacci sequence given in Section 2, the analysis is significantly different – each element in the set L is a nonnegative integer linear combination of the first three terms of L. Hence, for each $k \geq 3$, we show that $\mathfrak{L}_{2k-1}a + \mathfrak{L}_{2k}b$ belongs to $\Gamma\big(\{a, a+3b, 4a+7b\}\big)$. Write $\mathfrak{L}_{2k} = 3q + r$, with $r \in \{0,1,2\}$. Since $\mathfrak{L}_{2k} \in \{18, 47, \ldots\}$, $q \geq 15$ if $r \neq 0$. If $r = 0$, then $\mathfrak{L}_{2k-1}a + \mathfrak{L}_{2k}b = q(a+3b) + \big(\mathfrak{L}_{2k-1} - q\big)a$. If $r = 1$, then $\mathfrak{L}_{2k-1}a + \mathfrak{L}_{2k}b = (q-2)(a+3b) + (4a+7b) + \big(\mathfrak{L}_{2k-1} - q - 2\big)a$. If $r = 2$, then $\mathfrak{L}_{2k-1}a + \mathfrak{L}_{2k}b = (q-4)(a+3b) + 2(4a+7b) + \big(\mathfrak{L}_{2k-1} - q - 4\big)a$. Since $2\mathfrak{L}_{2k-1} > \mathfrak{L}_{2k-1} + \mathfrak{L}_{2k-2} = \mathfrak{L}_{2k} = 3q + r$, we have $\mathfrak{L}_{2k-1} > \frac{3}{2}q > q$. Moreover, $\mathfrak{L}_{2k-1} \geq q + 4$ when $r \neq 0$ since $q \geq 15$ in these cases. Therefore

$$\Gamma\big(L\big) = \Gamma\big(\{a, a+3b, 4a+7b\}\big). \tag{9}$$

Hence $\mathtt{g}(L)$ denotes the largest integer N such that the equation

$$ax_0 + (a+3b)x_1 + (4a+7b)x_2 = N \tag{10}$$

has no solution in nonnegative integers x_0, x_1, x_2, and $\mathtt{n}(L)$ the number of such N.

It is trivial that $\Gamma^c(L) = \emptyset$ if $1 \in L$, and that $\Gamma^c(L) = \{1, 3, 5, \ldots, m-2\}$ if $2 \in L$ and m (> 1) is the least odd integer in L. Therefore we may henceforth assume that $1 \notin L$ and $2 \notin L$.

The terms $\mathfrak{L}_2 = 3$ and $\mathfrak{L}_4 = 7$ play a significant part in the determination of the minimum integer in each residue class modulo a. Since $3(x+7m) + 7(y-3m) = 3x + 7y$ for any $m \in \mathbb{Z}$, we note that $\Gamma\big(\{3,7\}\big) = \big\{3x + 7y : x \geq 0, y \in \{0,1,2\}\big\}$; henceforth we use this representation.

Proposition 4. *Let a, b be positive integers, with $a \geq 3$ and $\gcd(a,b) = 1$. Let*

$$L = \big\{a, \mathfrak{L}_1 a + \mathfrak{L}_2 b, \mathfrak{L}_3 a + \mathfrak{L}_4 b, \ldots, \mathfrak{L}_{2k-1} a + \mathfrak{L}_{2k} b\big\}.$$

For each $y \in \{1, \ldots, a-1\}$, let $t = t_y$ be the least nonnegative integer such that $y + ta = 3r_y + 7s_y \in \Gamma\big(\{3,7\}\big)$, with $s_y \in \{0,1,2\}$. Then the minimum integer in $\Gamma(L) \cap (by)$ is given by

$$\mathbf{m}_{by} = a(r_y + 4s_y) + b(3r_y + 7s_y),$$

except when $(a, y) \in \{(4,3), (5,1), (5,3)\}$.

Proof. Let $\mathbf{m}_{by}$ denote the least positive integer in the class (by) modulo a. By (9) and (10), $\mathbf{m}_{by}$ is the least positive integer of the form $ax + by$ with $x = x_0 + 3x_1 + 4x_2$ and $y = 3x_1 + 7x_2$. For $y \in \{1, \ldots, a-1\}$, we must therefore minimize $x_0 + 3x_1 + 4x_2$ subject to $3x_1 + 7x_2 \equiv y \pmod{a}$, with each $x_i \geq 0$. Since x_0 is not a part of the constraint, the objective function is actually $3x_1 + 4x_2$. Since the transformation $(x_1, x_2) \mapsto (x_1 + 7, x_2 - 3)$ leaves $3x_1 + 7x_2$ fixed but reduces $x_1 + 4x_2$ (by 5), we must choose $x_2 \in \{0,1,2\}$ among the solutions to $3x_1 + 7x_2 = y + at$.

Suppose $y + at = 3x_1 + 7x_2 \in \Gamma\big(\{3,7\}\big)$, with $y \in \{1,\ldots,a-1\}$. We claim that $y+a(t+1) = 3x_1' + 7x_2' \in \Gamma\big(\{3,7\}\big)$, except when $(a,y) \in \{(4,3),(5,1),(5,3)\}$. The claim obviously holds when $a \in \Gamma\big(\{3,7\}\big)$. Since $a \geq 3$, the only cases that remain are when $a \in \{4,5,8,11\}$. When $a \in \{4,5\}$, the only exceptions are provided by $(a,y) \in \{(4,3),(5,1),(5,3)\}$. Since $y + a(t+1) \geq 1 + 2\cdot 8 > \mathtt{g}(3,7)$ for $a \in \{8,11\}$, the claim holds in this case.

Suppose that both $y + at = 3x_1 + 7x_2$ and $y + a(t+1) = 3x_1' + 7x_2'$ belong to $\Gamma\big(\{3,7\}\big)$; hence $(a,y) \notin \{(4,3),(5,1),(5,3)\}$. Then $a = 3(x_1' - x_1) + 7(x_2' - x_2)$. Now $(x_1' - x_1) + 4(x_2' - x_2) \geq 0$ if and only if $3(x_1' - x_1) + 12(x_2' - x_2) \geq 0$, which is the same as $a + 5(x_2' - x_2) \geq 0$. Therefore

$$x_1' + 4x_2' \geq x_1 + 4x_2 \text{ if and only if } a + 5(x_2' - x_2) \geq 0. \tag{11}$$

We show that $x_1' + 4x_2' \geq x_1 + 4x_2$ holds in this case. By (11), we need to consider only the case when $x_2' - x_2 \in \{-1,-2\}$. If $a \equiv 0 \pmod 3$, then $x_2' - x_2 \equiv 0 \pmod 3$. Hence $x_2 = x_2'$, and we are done. If $a \equiv 1 \pmod 3$, then $x_2' - x_2 \equiv 1 \pmod 3$. Assume that $x_2' - x_2 = -2$. Then (11) holds for $a \geq 10$. Suppose $a = 4$ or 7, and fix $y \in \{1,\ldots,a-1\}$. Let t_y denote the least nonnegative integer such that $y + t_y a \in \Gamma\big(\{3,7\}\big)$. For $a = 4$ and $y \in \{1,2\}$, it is easy to check that $(t_1,t_2) = (2,1)$; in each case, $y + t_y a$ is a multiple of 3. Hence $x_2 = 0$ in both cases. For $a = 7$, $y + t_y a = y + 7t_y$ must be a multiple of 3; otherwise there would be a contradiction to the definition of t_y. Hence $x_2 = 0$ in these case too. But then $x_2' - x_2 = x_2 \neq -2$. This completes the argument for $a \equiv 1 \pmod 3$. If $a \equiv 2 \pmod 3$, then $x_2' - x_2 \equiv 2 \pmod 3$. Assume that $x_2' - x_2 = -1$. Then (11) holds for $a \geq 5$. Since $a \neq 2$, this completes the argument for $a \equiv 2 \pmod 3$.

This completes the proof of the proposition. $\square$

Proposition 5. *Let a,b be positive integers, with $a \geq 12$ and $\gcd(a,b) = 1$. Let*

$$L = \big\{a, \mathcal{L}_1 a + \mathcal{L}_2 b, \mathcal{L}_3 a + \mathcal{L}_4 b, \ldots, \mathcal{L}_{2k-1} a + \mathcal{L}_{2k} b\big\}.$$

Then

(i)

$$\mathtt{g}(L) = \begin{cases} a\left(\frac{1}{3}a + 6\right) + b(a+11) & \text{if } 3 \mid a; \\ \max\left\{a\left(\frac{1}{3}(a-1) + 4\right) + b(a+4), a\left(\frac{1}{3}(a-1) + 3\right) + b(a+11)\right\} & \text{if } 3 \mid (a-1); \\ a\left(\frac{1}{3}(a-2) + 5\right) + b(a+11) & \text{if } 3 \mid (a-2). \end{cases}$$

(ii) $\mathtt{n}\big(L\big)$ *satisfies the equation*

$$a\Big(\mathtt{n}\big(L\big) + \tfrac{a-1}{2}\Big) = (a+3b)R + (4a+7b)S,$$

where

$$(R,S) = \begin{cases} \left(\frac{1}{6}(a^2 - 3a), a\right) & \text{if } 3 \mid a; \\ \left(\frac{1}{6}(a^2 - 3a + 98), a - 7\right) & \text{if } 3 \nmid a. \end{cases}$$

Proof. Let

$$S_1 = \{1,2,4,5,8,11\}, \quad S_2 = \mathbb{N} \setminus S_1, \quad S_i(a) = S_i \cap \{1,\dots,a-1\}, i \in \{1,2\}.$$

(i) By Proposition 4, $\mathbf{m}_{b(y+3)} > \mathbf{m}_{by}$ whenever $y \in \Gamma\big(\{3,7\}\big)$ and $y+3 \le a-1$. Note that $n \in S_1$ if and only if $n \notin \Gamma\big(\{3,7\}\big)$. Hence $t_y > 0$ if and only if $y \in S_1(a)$. By Lemma 1 and Proposition 4, with $s_y = (y + t_y a) \bmod 3$ and $r_y = \frac{1}{3}(y + t_y a - 7s_y)$, we have

$$\begin{aligned} \mathtt{g}(L) &= \max_{y \in S_1(a)} \mathbf{m}_{by} - a \\ &= \max_{y \in S_1(a)} \big(a\,(r_y + 4s_y) + b\,(3r_y + 7s_y)\,\big) - a. \end{aligned}$$

For $a \ge 12$, $t_y = 1$ for $y \in S_1(a)$, so that $s_y = (y+a) \bmod 3$ and $r_y = \frac{1}{3}(y + a - 7s_y)$. Since $s_1 = s_4$, $r_1 < r_4$, and $s_2 = s_5 = s_8 = s_{11}$, $r_2 < r_5 < r_8 < r_{11}$, we have

$$\begin{aligned} \mathtt{g}(L) &= \max\Big\{a\big(r_4 + 4s_4\big) + b\big(3r_4 + 7s_4\big), a\big(r_{11} + 4s_{11}\big) + b\big(3r_{11} + 7s_{11}\big)\Big\} - a \\ &= \begin{cases} a\left(\frac{1}{3}a + 6\right) + b(a+11) & \text{if } 3 \mid a; \\ \max\left\{a\left(\frac{1}{3}(a-1) + 4\right) + b(a+4), a\left(\frac{1}{3}(a-1) + 3\right) + b(a+11)\right\} & \text{if } 3 \mid (a-1); \\ a\left(\frac{1}{3}(a-2) + 5\right) + b(a+11) & \text{if } 3 \mid (a-2). \end{cases} \end{aligned}$$

(ii) From Lemma 1 and Proposition 4,

$$a\Big(\mathtt{n}\big(L\big) + \tfrac{a-1}{2}\Big) = \sum_{y=1}^{a-1} \mathbf{m}_{by} = (a+3b)\sum_{y=1}^{a-1} r_y + (4a+7b)\sum_{y=1}^{a-1} s_y. \tag{12}$$

Recall from part (i) that

$$s_y = (y + t_y a) \bmod 3, \quad r_y = \frac{1}{3}(y + t_y a - 7s_y),$$

where t_y equals 1 when $y \in S_1(a)$, and 0 when $y \in S_2(a)$.

We observe that

$$\sum_{y=1}^{a-1} y \bmod 3 = \begin{cases} \frac{a}{3}(1+2) = a & \text{if } 3 \mid a; \\ \frac{a-1}{3}(1+2) = a-1 & \text{if } 3 \mid (a-1); \\ \frac{a-2}{3}(1+2) + 1 = a-1 & \text{if } 3 \mid (a-2). \end{cases}$$

It is also easy to verify that $\sum_{y \in S_1(a)} (y+a) \bmod 3 = \sum_{y \in S_1(a)} (y \bmod 3) - 6$

when $3 \nmid a$. Therefore

$$S = \sum_{y=1}^{a-1} s_y = \sum_{y \in S_1(a)} (y+a) \bmod 3 + \sum_{y \in S_2(a)} y \bmod 3$$
$$= \begin{cases} \sum_{y=1}^{a-1} y \bmod 3 = a & \text{if } 3 \mid a; \\ \sum_{y=1}^{a-1} y \bmod 3 - 6 = a - 7 & \text{if } 3 \nmid a. \end{cases}$$

Hence

$$\begin{aligned} 3R = 3\sum_{y=1}^{a-1} r_y &= \sum_{y \in S_1(a)} (y + a - 7s_y) + \sum_{y \in S_2(a)} (y - 7s_y) \\ &= \sum_{y=1}^{a-1} y + a \sum_{y \in S_1(a)} 1 - 7\sum_{y=1}^{a-1} s_y \\ &= \frac{1}{2}a(a-1) + 6a - 7S \\ &= \begin{cases} \frac{1}{2}(a^2 - 3a) & \text{if } 3 \mid a; \\ \frac{1}{2}(a^2 - 3a + 98) & \text{if } 3 \nmid a. \end{cases} \end{aligned}$$

□

Remark 2. *Theorem 5 applies to $a \geq 12$. We list the values of $\mathtt{g}(L)$ and $\mathtt{n}(L)$ for $a \in \{3, \ldots, 11\}$.*

a	$\mathtt{g}(L)$	$\mathtt{n}(L)$
3	$14b + 21$	$7b + 11$
4	$9b + 8$	$\frac{1}{2}(9b + 9)$
5	$\max\{7b + 15, 9b + 10\}$	$5b + 8$
6	$17b + 48$	$\frac{1}{2}(17b + 49)$
7	$18b + 35$	$9b + 18$
8	$13b + 40$	$\frac{1}{2}(15b + 43)$
9	$20b + 81$	$10b + 41$
10	$21b + 60$	$\frac{1}{2}(21b + 71)$
11	$19b + 77$	$10b + 40$

4. Concluding Remarks

The Fibonacci and Lucas sequences satisfy the second-order recurrence equation $u_n = u_{n-1} + u_{n-2}$ for $n \geq 2$, and therefore are given by the formula $u_n = A\alpha^n + B\beta^n$. The constants A and B are determined by the initial terms u_0 and u_1 of the sequence. The problem of finding exact values of the functions $\mathtt{g}$ and $\mathtt{n}$ for sets related to any such sequence is possible by methods similar to those adopted in this article. In fact, it would be interesting to consider sets that arise from linear recurrences of any order in some natural way. We close by formulating one of these general problems.

Problem 1. *Let* $\{\mathcal{U}_n\}_{n\geq 0}$ *be a second-order recurrence equation given by* $\mathcal{U}_n = c_1\mathcal{U}_{n-1} + c_2\mathcal{U}_{n-2}$ *for* $n \geq 2$*. For positive and coprime integers* a, b*, let*

$$S = \big\{a, \mathcal{U}_1 a + \mathcal{U}_2 b, \mathcal{U}_3 a + \mathcal{U}_4 b, \ldots, \mathcal{U}_{2k-1} a + \mathcal{U}_{2k} b\big\}.$$

Give exact values for $\mathtt{g}(S)$ *and* $\mathtt{n}(S)$*.*

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