A Curious Property of the Decimal Expansion of Reciprocals of Primes*

Amitabha Tripathi†

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Abstract

For prime $p \neq 2, 5$, the decimal expansion of $1/p$ is purely periodic. For those prime p for which the length of the period is even, if we break up the digits in the periodic part into two equal parts and add, we get a number all of whose digits are 9. We provide a self-contained and simple proof of this old result of E. Midy.

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The decimal expansion of $1/p$ when p is prime has aroused much renewed curiosity following an article by Brian Ginsberg [2], then an undergraduate student at Yale University, in the *College Mathematics Journal* in 2004. Ginsberg extended a curious result of E. Midy [9] about the decimal expansion of $1/p$ in 1836 when p is a prime distinct from 2 and 5. As often happens in such cases, this result was rediscovered several times, for instance in [6] and [11]. Midy's Theorem has recently been the subject of much discussion following the article by Ginsberg; for example, see ([3], [7], [8]).

Before we state and prove Midy's Theorem, let us look at some examples. Consider, for instance, the decimal expansions of $1/p$ for $p = 7$, 13 and 19:

$$\begin{aligned} \tfrac{1}{7} &= 0.\overline{142857} \\ \tfrac{1}{13} &= 0.\overline{076923} \\ \tfrac{1}{19} &= 0.\overline{052631578947368421} \end{aligned}$$

We observe that the decimal expansion in each case is purely periodic, and the length of the period is even. While the first part is always true for primes $p \neq 2, 5$, the second part does not always hold; for example, $\frac{1}{3} = 0.\overline{3}$ has period length 1. Midy's result is about primes p for which the decimal expansion of

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†Indian Institute of Technology, Delhi

$1/p$ has even period length. In such cases, if we break up the repeating digits into two equal blocks and add we find that we get a number with only 9's:

$$142 + 857 = 999; \quad 076 + 923 = 999; \quad 052\,631\,578 + 947\,368\,421 = 999\,999\,999.$$

It is the purpose of this short note to explain this rather curious but not so well known result of Midy. However, before the main result, we give the following characterization of the decimal expansion of rational numbers which is well known (see [5]). The period length of the digits in the expansion of $1/p$ is linked to the following definition.

Definition 1. *Let $m \in \mathbb{N}$. If $\gcd(a, m) = 1$, the least positive integer h such that $m \mid (a^h - 1)$ is called the order of a mod m and denoted by $ord_m\, a$.*

The existence of $ord_m\, a$ is a consequence of *Euler's Theorem*, which states that $m \mid (a^{\phi(m)} - 1)$ where

$$\phi(m) = m \prod_{p|m} \left(1 - \frac{1}{p}\right),$$

with the empty product taken as 1. This is simply *Lagrange's Theorem* applied to the multiplicative group of units in $\mathbb{Z}_m$, whose order is defined to be $\phi(m)$. We are now in a position to explain the nature of the decimal expansion of rational numbers.

Lemma 1. *Let a, n be positive integers, with $n > 1$ and $\gcd(a, n) = 1$. Let $n = 2^\alpha 5^\beta m$, with $\gcd(m, 10) = 1$. Then the decimal expansion of a/n is recurring if and only if $m > 1$. Moreover, if $m > 1$, the length of the non-recurring part is $\max\{\alpha, \beta\}$ while that of the recurring part is $ord_m\, 10$.*

Proof. The decimal expansion of a/n terminates if and only if a/n is of the form $A/10^k$ for some A, k, which is true if and only if $n \mid 10^k$. The last statement holds precisely when $m = 1$, and this completes the first part of the lemma.

Assume $n = 2^\alpha 5^\beta m$, with $\gcd(m, 10) = 1$ and $m > 1$. If we reduce each term of the infinite sequence $\{a10^i\}_{i \geq 1}$ mod n, by *Pigeonhole Principle*, two terms must be equal. If we choose the first such pair, say $a10^s$ and $a10^{s+\ell}$, then

$$10^s \equiv 10^{s+\ell} \pmod{n}$$

since $\gcd(a, n) = 1$. Hence

$$10^\ell \equiv 1 \pmod{m}$$

since $\gcd(m, 10) = 1$. Since the congruence implies that $a10^{s+k} \equiv a10^{s+\ell+k} \pmod{n}$ for each $k \geq 0$, the length of the period equals $\ell = ord_m\, 10$.

From the first congruence we see that both 2^α and 5^β must divide 10^s since each is coprime to $10^\ell - 1$. The smallest such s equals $\max\{\alpha, \beta\}$, and this represents the length of the non-recurring part. ■

Before we look at the proof of Midy's Theorem, let us take another look at the examples we considered at the beginning in light of Lemma 1. The decimal expansion of $1/p$ when $\gcd(p, 10) = 1$ is purely periodic, and the length of the period is the least positive integer h such that $p \mid (10^h - 1) = \underbrace{9\ldots9}$ consisting of h digits. All we can say for sure is that $ord_p\, 10 = h$ must divide $\phi(p) = p-1$, but it is not possible to determine h more explicitly. **It is a famous conjecture of** *C. F. Gauss* **that there are infinitely many primes p for which the decimal expansion of $1/p$ has period length $p - 1$.**

Theorem 1. (Midy)

Suppose p is a prime, $p \neq 2, 5$, for which $ord_p\, 10 = 2d$ for some $d \geq 1$. Then

$$\tfrac{1}{p} = 0.\overline{a_1\, a_2\, \ldots\, a_d\, a_{d+1}\, a_{d+2}\, \ldots\, a_{2d}},$$

where $a_k + a_{d+k} = 9$ for $1 \leq k \leq d$.

Proof. We know from Lemma 1 that the decimal expansion of $1/p$ has a purely recurring expansion, and the length of the period equals $ord_p\, 10 = 2d$. Let us denote by A_k the k-digit number $a_1 a_2 \ldots a_k$ for $k \geq 1$. Since A_k is the quotient of 10^k by p, we can write

$$10^k = pA_k + r_k, \tag{1}$$

with $1 \leq r_k < p$. Since $1/p$ is the sum of an infinite geometric series with first term $A_{2d}/10^{2d}$ and common ratio $1/10^{2d}$, we have

$$10^{2d} = pA_{2d} + 1. \tag{2}$$

Now $ord_p\, 10 = 2d$ implies $p \mid (10^d + 1)(10^d - 1)$, so that p must divide one of $10^d - 1$, $10^d + 1$. But $p \mid (10^d - 1)$ is not possible since $ord_p\, 10 = 2d$; so must have $10^d \equiv -1 \pmod{p}$. Thus

$$10^d = pA_d + (p - 1). \tag{3}$$

If we denote by A'_k the $(2d - k)$-digit number $a_{k+1} a_{k+2} \ldots a_{2d}$ for $k \geq 1$, then $A_{2d} = A_d \cdot 10^d + A'_d$. From (2) and (3) we have

$$A_d + A'_d = A_{2d} - (10^d - 1)A_d = \frac{\big(10^{2d} - 1) - (10^d - 1)(10^d - (p - 1)\big)}{p} = 10^d - 1.$$

Hence the theorem. ■

We close this article by remarking that the result of Midy extends to the decimal expansion of a/p for $1 < a < p$. Lemma 1 implies that the decimal expansion of a/p is purely periodic, with period length $ord_p\, 10$. If $ord_p\, 10 = p - 1$, then the decimal digits in the expansion of a/p are got by a cyclic shift of those of $1/p$. For example, cyclic shifts of the decimal digits in the expansion of $\frac{1}{7} = 0.\overline{142857}$ accounts for

$$\tfrac{3}{7} = 0.\overline{428571},\ \tfrac{2}{7} = 0.\overline{285714},\ \tfrac{6}{7} = 0.\overline{857142},\ \tfrac{4}{7} = 0.\overline{571428},\ \tfrac{5}{7} = 0.\overline{714285}.$$

If $ord_p\, 10 = \ell < p-1$, then $\ell \mid (p-1)$ and cyclic shifts of the decimal digits in the expansion of $1/p$ accounts for ℓ of the fractions a/p. The fractions a/p for $1 \le a \le p-1$ are partitioned into $(p-1)/\ell$ classes, with the digits in each class obtained from one another by cyclic shifts of the decimal digits. For example, cyclic shifts of the decimal digits of $\frac{1}{13} = 0.\overline{076923}$ accounts for the fractions

$$\tfrac{10}{13} = 0.\overline{769230},\ \tfrac{9}{13} = 0.\overline{692307},\ \tfrac{12}{13} = 0.\overline{923076},\ \tfrac{3}{13} = 0.\overline{230769},\ \tfrac{4}{13} = 0.\overline{307692}.$$

The remaining fractions,

$$\tfrac{7}{13} = 0.\overline{538461},\ \tfrac{5}{13} = 0.\overline{384615},\ \tfrac{11}{13} = 0.\overline{846153},\ \tfrac{6}{13} = 0.\overline{461538},\ \tfrac{8}{13} = 0.\overline{615384},$$

result from cyclic shifts of $\frac{2}{13} = 0.\overline{153846}$. In general, if we start with the decimal expansion of a/p, the cyclic shifts results in the decimal expansion of $10a/p$, $10^2a/p$, ..., $10^{\ell-1}a/p$, where ℓ is the periodic length of the expansion of a/p.

Theorem 1 easily extends to any a/p, where $1 < a < p$. Indeed, assume as in Theorem 1, that p is a prime, $p \ne 2, 5$, for which $ord_p\, 10 = 2d$ for some $d \ge 1$. Then

$$\tfrac{a}{p} = 0.\overline{b_1\, b_2\, \ldots\, b_d\, b_{d+1}\, b_{d+2}\, \ldots\, b_{2d}},$$

where $b_k + b_{d+k} = 9$ for $1 \le k \le d$.

To see this, write B_k for the k-digit number $b_1 b_2 \ldots b_k$ for $k \ge 1$. Thus

$$a \cdot 10^k = pB_k + s_k, \tag{4}$$

with $1 \le s_k < p$. Hence

$$a \cdot 10^{2d} = pB_{2d} + a, \quad a \cdot 10^d = pB_d + (p-a). \tag{5}$$

If we denote by B'_k the $(2d-k)$-digit number $b_{k+1}b_{k+2}\ldots b_{2d}$ for $k \ge 1$, then $B_{2d} = B_d \cdot 10^d + B'_d$. From (5) we have

$$B_d + B'_d = B_{2d} - (10^d - 1)B_d = \frac{a\big(10^{2d}-1\big) - (10^d-1)\big(a \cdot 10^d - (p-a)\big)}{p} = 10^d - 1.$$

Hence the extension to any a/p.

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About the author:

I enjoy lecturing on mathematical topics that are not routine, in particular mathematical problem solving. Outside of mathematics, I enjoy playing table tennis, working on crosswords and sudoku, listening to music, and when given an opportunity, to demonstrate my computational abilities.

Amitabha Tripathi

Department of Mathematics,
Indian Institute of Technology,
Hauz Khas, New Delhi 110016.
atripath@maths.iitd.ac.in