EXTENSIONS OF MIDY'S THEOREM FOR PERIODIC DECIMALS

Sakshi Dang[1]
Department of Mathematics, Indian Institute of Technology, Powai, Mumbai, India
204093005@iitb.ac.in

Saraswati Nanoti[2]
Discipline of Mathematics, Indian Institute of Technology, Palaj, Gandhinagar, Gujarat, India
nanoti_saraswati@iitgn.ac.in

Amitabha Tripathi[3]
Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi, India
atripath@maths.iitd.ac.in

Received: 5/16/20, Accepted: 2/17/21, Published: 3/23/21

Abstract

The decimal expansion of reduced rational number a/n is purely recurring precisely when $\gcd(n, 10) = 1$. For any reduced fraction a/n, with $\gcd(n, 10) = 1$, having periodic length $L = b\ell$, partition the periodic part into b blocks, each of length ℓ, and let $S(a, n; \ell, b)$ denote the sum of the b blocks. We show that

$$S(a, n; \ell, b) = (k + qd) \cdot \tfrac{10^\ell - 1}{d},$$

where k, d can be determined from a, n, ℓ, b and $q \in \{0, \dots, b-1\}$. In particular, we show (i) $S(a, p; \ell, b) = 10^\ell - 1$ when $b \in \{2, 3\}$, (ii) $S(a, n; \ell, b) = \frac{b}{2}(10^\ell - 1)$ when b is even and $n \mid (10^{L/2} + 1)$, and determine (iii) $S(a, n; \ell, 2)$. We also characterize n for which $S(a, n; \ell, 2)$ equals either λ_ℓ, $1 \le \lambda \le 9$ or $\lambda_\ell + 9_\ell$, $1 \le \lambda \le 8$, where λ_ℓ is the ℓ-digit number $\lambda \dots \lambda$ for any $\lambda \in \{1, \dots, 9\}$. Our results contain the theorems of Midy and Ginsberg, and either contain or extend the result of several other authors.

1. Introduction

Midy [9] discovered the result that goes by his name in 1836, but it was probably due to Dickson [1] that the result became known to people in mathematical circles.

[1]This work was done while visiting Department of Mathematics, IIT Delhi.
[2]This work was done while visiting Department of Mathematics, IIT Delhi.
[3]Corresponding Author

Because of the approachable yet mysterious nature of the result, it found a place in several well known works that dealt with the treatment of Mathematics largely from the point of view of entertainment; for instance, in [4, 11]. A recent paper of Lewittes [7] traces some of the history behind Midy's intriguing result. Interest in Midy's theorem has been sporadic until very recently; [6, 10], to name a few. A resurgence of interest in Midy's theorem and attempts to generalize it may be directly attributed to an extension of the original result by Ginsberg [2], resulting in significant extensions by Gupta and Sury [3], Lewittes [7], and Martin [8], among others.

Let $a, n \in \mathbb{N}$, $n > 1$, $1 \le a < n$, and $\gcd(a, n) = 1$. Let $n = 2^\alpha \cdot 5^\beta \cdot m$, where $\gcd(m, 10) = 1$. Then the decimal expansion of a/n is given by

$$\frac{a}{n} = 0.c_1 \, \ldots \, c_\gamma \, \overline{c_{\gamma+1} \, \ldots \, c_{\gamma+L}}, \tag{1}$$

where $\gamma = \max\{\alpha, \beta\}$ and $L = \operatorname{ord}_m 10$ is the multiplicative order of 10 modulo m, that is, the least positive integer k satisfying $10^k \equiv 1 \pmod{m}$. In particular, the decimal expansion of a/n is purely recurring if and only if $\gcd(n, 10) = 1$. All this is well known; see, for instance [5].

We consider the decimal expansion of a/n, where $\gcd(10a, n) = 1$ throughout the rest of this article. In this case, $\alpha = \beta = \gamma = 0$ and $m = n$ in Equation (1), and so

$$\frac{a}{n} = 0.\overline{c_1 \, \ldots \, c_L}. \tag{2}$$

Let $B(a, n)$ denote the smallest repeating block of digits in decimal expansion of a/n:

$$B(a, n) = c_1 \, \ldots \, c_L. \tag{3}$$

The number of digits in $B(a, n)$ is called the *period length* of a/n.

Suppose that L is divisible by b, that is, the L-length period $B(a, n)$ can be divided into b blocks, each of length ℓ; thus $L = b\ell$. Since a/n is the sum of an infinite geometric progression with first term $B(a, n)/10^L$ and common ratio $1/10^L$. We have

$$n \cdot B(a, n) = a(10^L - 1). \tag{4}$$

We divide $B(a, n)$ into b subblocks, each of length ℓ:

$$\begin{array}{rcl}
B_1(a, n; \ell, b) & = & c_1 \, \ldots \, c_\ell, \\
B_2(a, n; \ell, b) & = & c_{\ell+1} \, \ldots \, c_{2\ell}, \\
B_3(a, n; \ell, b) & = & c_{2\ell+1} \, \ldots \, c_{3\ell}, \\
\vdots & & \vdots \\
B_b(a, n; \ell, b) & = & c_{(b-1)\ell+1} \, \ldots \, c_{b\ell}.
\end{array}$$

Let $S(a,n;\ell,b)$ denote the sum of these blocks:

$$S(a,n;\ell,b) = B_1(a,n;\ell,b) + \cdots + B_b(a,n;\ell,b). \tag{5}$$

The organization of this article is as follows. In Section 2, we give a formula to compute the sums $S(a,n;\ell,b)$, defined by Equation (5). As a consequence, we prove the theorems of Midy [9] and Ginsberg [2], that $S(1,p;\ell,b) = 10^\ell - 1$ for prime $p > 5$ and $b = 2$, 3, respectively. In Section 3, we deal with the case where b is even and n divides $10^{b\ell/2} + 1$. We note that this case applies when $n = p^\alpha$, p prime and L is even. In Section 4, we give an explicit formula for $S(a,n;\ell,2)$. We also characterize n for which $S(a,n;\ell,2)$ equals either λ_ℓ, $1 \le \lambda \le 9$ or $\lambda_\ell + 9_\ell$, $1 \le \lambda \le 8$, where λ_ℓ is the ℓ-digit number $\lambda \ldots \lambda$ for any $\lambda \in \{1,\ldots,9\}$. Some results by previous authors, like Lewittes [7] and Martin [8], may either be deduced from our results, or generalized.

2. The General Case

We consider the decimal expansion of a/n, where $\gcd(a,n) = \gcd(n,10) = 1$. Such decimal expansions are purely recurring. We denote by L the length of the recurring part, and break up the recurring part into b blocks each of length ℓ. We denote by $S(a,n;\ell,b)$ the sum of these b numbers each of ℓ digits. Theorem 1 shows that

$$S(a,n;\ell,b) = \left(\tfrac{k}{d} + q\right)(10^\ell - 1),$$

where k, d may be computed from the given parameters a, n, ℓ, b, and $q \in \{0,\ldots,b-1\}$. For prime $n = p$, the theorems of Midy and Ginsberg correspond to $b = 2$ and $b = 3$, respectively, and follow easily from Theorem 1; see Corollary 1. Whereas Midy's theorem applies to any a coprime to $10n$, Ginsberg proved his result only in the case $a = 1$. We illustrate the results in Theorem 1 and Corollary 1 by numerical examples.

Theorem 1. *Let $a, n \in \mathbb{N}$, with $1 \le a < n$ and $\gcd(10a,n) = 1$. Let $\operatorname{ord}_n 10 = L$, and let $b, \ell \in \mathbb{N}$ such that $L = b\ell$ and $b > 1$. Define*

$$N(\ell,b) = \frac{10^L - 1}{10^\ell - 1}, \quad g = \gcd\big(n, N(\ell,b)\big), \quad d = \frac{n}{g}.$$

Then

$$S(a,n;\ell,b) = (k + qd) \cdot \frac{10^\ell - 1}{d},$$

where

$$k \equiv \frac{N(\ell,b)}{g} \cdot a \pmod{d}, k \in \{0,\ldots,d-1\}, \text{ and } q \in \{0,\ldots,b-1\}.$$

In particular, $S(a,n;\ell,b) = q(10^\ell - 1)$ for some $q \in \{1,\ldots,b-1\}$ if $d = 1$.

Proof. Let $\frac{a}{n} = 0.\overline{c_1 \ldots c_L}$. For each $t \in \mathbb{N}$, let A_t denote the t-digit number $c_1 \ldots c_t$. Then

$$a \cdot 10^t = n \cdot A_t + r_t,\ 1 \le r_t < n, \tag{6}$$

and

$$B_t(a,n;\ell,b) = A_{t\ell} - 10^\ell \cdot A_{(t-1)\ell}, 2 \le t \le \ell, \text{ with } B_1(a,n;\ell,b) = A_\ell. \tag{7}$$

In particular, $r_{b\ell} = a$. Substituting Equation (6) in Equation (5) gives

$$S(a,n;\ell,b) = \frac{1}{n}\left(\sum_{t=1}^{b} r_{t\ell}\right)(10^\ell - 1). \tag{8}$$

Note that n does not divide $10^\ell - 1$ but may divide $\sum_{t=1}^{b} r_{t\ell}$.

Putting $t = \ell, 2\ell, 3\ell, \ldots, b\ell$ in Equation (6) and adding gives

$$a \cdot N(\ell,b) \cdot 10^\ell = n\sum_{i=1}^{b} A_{i\ell} + \sum_{i=1}^{b} r_{i\ell}. \tag{9}$$

Since g divides both $N(\ell,b)$ and n, we have g divides $\sum_{i=1}^{b} r_{i\ell}$. Dividing the quotient $\sum_{i=1}^{b} r_{i\ell}/g$ by d allows us to write

$$\frac{\sum_{i=1}^{b} r_{i\ell}}{g} = qd + k \tag{10}$$

where $q \in \mathbb{Z}_{\ge 0}$ and $k \in \{0,\ldots,d-1\}$. Combining Equation (8) and Equation (10), and using $n = dg$ gives

$$S(a,n;\ell,b) = (k + qd) \cdot \frac{10^\ell - 1}{d}. \tag{11}$$

We note that

$$\frac{n}{g} = d \text{ divides } 10^\ell - 1 \tag{12}$$

follows from

$$\frac{N(\ell,b)}{g}(10^\ell - 1) = \frac{10^L - 1}{n} \cdot \frac{n}{g} \text{ and } g = \gcd\big(n, N(\ell,b)\big).$$

Dividing Equation (9) throughout by g, using Equation (10), and reducing modulo d gives

$$k \equiv \frac{N(\ell,b)}{g} \cdot a \pmod{d}.$$

Since $1 \le r_t < n$ from Equation (6), Equation (10) gives

$$0 \le qd \le qd + k = \frac{\sum_{i=1}^{b} r_{i\ell}}{g} < \frac{nb}{g} = bd.$$

Hence $q \in \{0,\ldots,b-1\}$. □

Example 1.

$$\tfrac{46}{561} = 0.\overline{08199643493761 14}.$$

a	n	L	b	ℓ	$N(\ell,b)$	g	d	k	q	$S(a,n;\ell,b)$
46	561	16	2	8	10^8+1	17	33	19	0	$\frac{19}{33}(10^8-1)$
			4	4	$10^{12}+10^8+10^4+1$			5	2	$\frac{5}{33}(10^4-1)+2(10^4-1)$
			8	2	$10^{14}+10^{12}+\cdots+1$			10	3	$\frac{10}{33}(10^2-1)+3(10^2-1)$

Table 1: Examples for Theorem 1

Corollary 1 ([2, 9])**.** *Let p be a prime, $p>5$. If $\operatorname{ord}_p 10 = L = b\ell$, where $b \in \{2,3\}$, then*

$$S(a,p;\ell,b) = 10^\ell - 1,$$

where $a \in \{1,\ldots,p-1\}$ when $b=2$ and $a=1$ when $b=3$.

Proof. We use the notations of Theorem 1. Since $\operatorname{ord}_p 10 = L$, $p \mid (10^L-1)$ and $p \nmid (10^\ell - 1)$. Hence $p \mid N(\ell,b)$, so that $g=p$ and $d=1$. The particular case of Theorem 1 shows $S(a,p;\ell,b) = q(10^\ell-1)$ for $q \in \{1,2,3,\ldots,b-1\}$. When $b=2$, $q=1$, and this is Midy's theorem. When $b=3$, $q=1$ or 2. By Equation (10) in Theorem 1, $q = \big(r_\ell + r_{2\ell} + r_{3\ell}\big)/p$. Since $r_\ell, r_{2\ell} \in \{1,\ldots,p-1\}$, $r_{3\ell} = a = 1$, and $p \mid (r_\ell + r_{2\ell} + r_{3\ell})$, it follows that $q=1$. This is Ginsberg's theorem. $\square$

Example 2.

$$\tfrac{4}{19} = 0.\overline{210526315789473684}.$$

a	n	L	b	ℓ	$N(\ell,b)$	g	d	k	q	$S(a,n;\ell,b)$
4	19	18	2	9	10^9+1	19	1	0	1	10^9-1
			3	6	$10^{12}+10^6+1$	19	1	0	1	10^6-1
			6	3	$10^{15}+10^{12}+10^9+\cdots+1$	19	1	0	3	$3(10^3-1)$
			9	2	$10^{16}+10^{14}+10^{12}+\cdots+1$	19	1	0	4	$4(10^2-1)$

Table 2: Examples for Corollary 1

3. The Case n Divides $10^{b\ell/2}+1$, b Even

In this section we evaluate the sum $S(a,n;\ell,b)$ when $n \mid (10^{L/2}+1)$ and b is even. One of the special cases where this applies is when $n = p^\alpha$ is a prime power, with $p \neq 2, 5$. Although the antecedents of a result of Lewittes [7, Theorem 1] seem to be different from our result in this section (Theorem 2), they are actually equivalent. We show this equivalence in Theorem 3. We close this section with numerical examples.

Theorem 2. *Let $a, n \in \mathbb{N}$, with $1 \leq a < n$ and $\gcd(10a, n) = 1$. Suppose that $\operatorname{ord}_n 10 = b\ell$ with b even. If $n \mid (10^{b\ell/2} + 1)$, then*

$$S(a, n; \ell, b) = \frac{b}{2}(10^{\ell} - 1).$$

Proof. We use the notations of Theorem 1. Putting $t = i\ell$ and $t = \left(i + \frac{b}{2}\right)\ell$ in Equation (6), adding and reducing modulo n gives

$$a \cdot 10^{i\ell} \cdot (1 + 10^{b\ell/2}) \equiv r_{i\ell} + r_{(i+\frac{b}{2})\ell} \pmod{n}. \tag{13}$$

Thus $n \mid \left(r_{i\ell} + r_{(i+\frac{b}{2})\ell}\right)$, and since $0 < r_{i\ell} + r_{(i+\frac{b}{2})\ell} < 2n$, we have $r_{i\ell} + r_{(i+\frac{b}{2})\ell} = n$. Therefore

$$\sum_{t=1}^{b} r_{t\ell} = \sum_{1=1}^{b/2} \left(r_{i\ell} + r_{(i+\frac{b}{2})\ell}\right) = \sum_{1=1}^{b/2} n = \frac{bn}{2}.$$

Substituting in Equation (8) gives the desired sum. □

Example 3.

$\frac{5}{61} = 0.\overline{081967213114754098360655737704918032786885245901639344262295}.$

a	n	L	b	ℓ	$S(a,n;\ell,b)$
5	61	60	2	30	$\frac{2}{2}(10^{30}-1)$
			4	15	$\frac{4}{2}(10^{15}-1)$
			6	10	$\frac{6}{2}(10^{10}-1)$
			10	6	$\frac{10}{2}(10^{6}-1)$
			12	5	$\frac{12}{2}(10^{5}-1)$
			20	3	$\frac{20}{2}(10^{3}-1)$
			30	2	$\frac{30}{2}(10^{2}-1)$

Table 3: Examples for Theorem 2

The antecedents of the following result (Theorem 3) that appear in [7] is equivalent to the antecedents of Theorem 2.

Theorem 3 ([7])**.** *Let $a, n \in \mathbb{N}$, with $1 \leq a < n$ and $\gcd(10a, n) = 1$. Suppose that $\operatorname{ord}_n 10 = b\ell$ with b even. If*

$$(10^{b\ell/2} - 1) \mid S\left(a, n; \tfrac{b\ell}{2}, 2\right),$$

then

$$S(a, n; \ell, b) = \frac{b}{2}(10^{\ell} - 1).$$

Proof. It suffices to prove the equivalence of the antecedents of Theorem 2 and Theorem 3, namely, the divisibility condition $(10^{b\ell/2}-1) \mid S\left(a,n;\frac{b\ell}{2},2\right)$ given in Theorem 3 is equivalent to the divisibility condition $n \mid (10^{b\ell/2}+1)$ in Theorem 2. We use the notations of Theorem 1.

If $n \mid (10^{b\ell/2}+1)$, then $g = \gcd\left(n, N\left(\frac{b\ell}{2},2\right)\right) = n$, so that $d=1$ and $k=0$. Hence $(10^{b\ell/2}-1) \mid S\left(a,n;\frac{b\ell}{2},2\right)$.

Conversely, suppose that $(10^{b\ell/2}-1) \mid S\left(a,n;\frac{b\ell}{2},2\right)$. By Theorem 1, $d \mid (k+qd)$ and so $d \mid k$. Hence $k=0$, so that $d \mid \frac{N\left(\frac{b\ell}{2},2\right)}{g} \cdot a$. Since $\gcd(a,d) = \gcd\left(\frac{N\left(\frac{b\ell}{2},2\right)}{g}, d\right) = 1$, we must have $d=1$. Therefore $n \mid (10^{b\ell/2}+1)$. $\square$

The theorem of Midy [9] is a direct consequence of Theorem 2. In fact, the "property of nines" may be generalized by the result in Corollary 2 to decimal expansions of a/p^α, p prime, $p \neq 2,5$ with even period length.

Corollary 2. *Let p be a prime with $p \neq 2,5$, and let a and α be positive integers such that $1 \le a < p^\alpha$ and $p \nmid a$. If $\mathrm{ord}_{p^\alpha} 10 = b\ell$, b even, then*

$$S(a,p^\alpha;\ell,b) = \frac{b}{2}(10^\ell - 1).$$

Proof. The condition $\mathrm{ord}_{p^\alpha} 10 = b\ell$ implies $p^\alpha \mid (10^{b\ell/2}+1)(10^{b\ell/2}-1)$. But then p^α must divide exactly one of $10^{b\ell/2} \pm 1$, since $\gcd(10^{b\ell/2}+1, 10^{b\ell/2}-1) = 1$. The possibility $p^\alpha \mid (10^{b\ell/2}-1)$ must be excluded due to the condition $\mathrm{ord}_{p^\alpha} 10 = b\ell$. Therefore $p^\alpha \mid (10^{b\ell/2}+1)$ and $S(a,p^\alpha;\ell,b) = \frac{b}{2}(10^\ell-1)$ by Theorem 2. $\square$

Example 4.

$$\tfrac{8}{49} = 0.\overline{163265306122448979591836734693877551020408}.$$

a	n	L	b	ℓ	$N(\ell,b)$	g	d	k	q	$S(a,n;\ell,b)$
8	49	42	2	21	$10^{21}+1$	49	1	0	1	$\frac{2}{2}(10^{21}-1)$
			6	7	$10^{35}+10^{28}+10^{21}+\cdots+1$	49	1	0	3	$\frac{6}{2}(10^{7}-1)$
			14	3	$10^{42}+10^{39}+10^{36}+\cdots+1$	49	1	0	7	$\frac{14}{2}(10^{3}-1)$

Table 4: Examples for Corollary 2

4. The Case $b = 2$

The main result of this section is the evaluation of the sum $S(a,n;\ell,2)$ in Theorem 4. Midy's theorem is the special case where $n = p$ is prime: $S(a,p;\ell,2) = 9_\ell$. We use Theorem 4 to characterize n for which $S(a,n;\ell,2)$ equals (i) λ_ℓ, $\lambda \in \{1,\ldots,9\}$, and (ii) $\lambda_\ell + 9_\ell$, $\lambda \in \{1,\ldots,8\}$; this is Theorem 5. In particular, Theorem 5 contains the following extension of Midy's theorem: For $1 \le a < n$, $\gcd(a,n) = 1$, $S(a,n;\ell,2) = 10^\ell - 1$ if and only if $n \mid (10^\ell + 1)$.

Martin [8] extended Midy's theorem to composite denominators by showing that $S(1,n;\ell,2)$ has the "property of nines" if and only if every pair of prime divisors of n are period-compatible. Two positive integers a and b are said to be *period-compatible* if the highest power of 2 in the prime factorization of period lengths of $1/a$ and $1/b$ is same. He also showed that $10^\ell - 1$ divides $S(1,n;\ell,2)$ under specific conditions. We show how these results of Martin in [8] can be deduced from Theorem 4 via Proposition 2 and Proposition 3 in Theorem 6. Wherever necessary, we exhibit numerical examples in support of our results.

Theorem 4. *Let $a, n \in \mathbb{N}$, with $1 \le a < n$ and $\gcd(10a,n) = 1$. Suppose that $\mathrm{ord}_n\, 10 = 2\ell$. Define*

$$g = \gcd\left(n, 10^\ell + 1\right), \quad d = \frac{n}{g}.$$

Then

$$S(a,n;\ell,2) = \begin{cases} k \cdot \dfrac{10^\ell - 1}{d} & \text{if } \ a < kg; \\ k \cdot \dfrac{10^\ell - 1}{d} + (10^\ell - 1) & \text{if } \ a > kg, \end{cases}$$

where k satisfies

$$k \equiv 2ag^{-1} \pmod{d},\ k \in \{0,\ldots,d-1\}.$$

Proof. We use the notations of Theorem 1. When $b = 2$, we have $N(\ell,2) = 10^\ell + 1$ and

$$S(a,n;\ell,2) = (k + qd) \cdot \frac{10^\ell - 1}{d},$$

where

$$k \equiv \frac{10^\ell + 1}{g} \cdot a \pmod{d} \ \text{ and } \ q \in \{0,1\}.$$

Note that $q = 0$ and $q = 1$ result in the expressions $k \cdot \frac{10^\ell - 1}{d}$ and $k \cdot \frac{10^\ell - 1}{d} + (10^\ell - 1)$ respectively, for $S(a,n;\ell,2)$. Thus, we need to show

- $\gcd(d,g) = 1$ and $N(\ell,2) = 10^\ell + 1 \equiv 2 \pmod{d}$, and
- the first case in the expression for $S(a,n;\ell,2)$ corresponds to $q = 0$ and the second case in the expression for $S(a,n;\ell,2)$ corresponds to $q = 1$.

Equation (12) in Theorem 1 shows that $d \mid (10^\ell - 1)$ and we know that $g \mid (10^\ell + 1)$. Since d and g are both odd, and any common divisor of d and g must divide $(10^\ell + 1) - (10^\ell - 1) = 2$, we have $\gcd(d,g) = 1$. The second part follows from $10^\ell \equiv 1 \pmod{d}$.

Putting $b = 2$ and $t = 2\ell = L$ in Equation (6) in Theorem 1, and reducing modulo n gives $r_{2\ell} \equiv a \pmod{n}$. Hence $r_{2\ell} = a$. From Equation (10) in Theorem 1 we have $r_\ell + r_{2\ell} = g(qd+k)$. Hence $r_\ell = g(qd+k) - a = qn + kg - a$. Since $0 < r_\ell < n$, $q = 0$ implies $a < kg$ and $q = 1$ implies $a > kg$. $\square$

Remark 1. We observe that with the notations of Theorem 4, $a \neq kg$ since $\gcd(a, n) = 1$.

Example 5.

$$\begin{aligned} \tfrac{25}{1547} &= 0.\overline{016160310277957336780866192630898513251454427925}, \\ \tfrac{389}{1547} &= 0.\overline{251454427925016160310277957336780866192630898513}. \end{aligned}$$

a	n	L	b	ℓ	$N(\ell, b)$	g	d	k	$S(a, n; \ell, b)$
25	1547	48	2	24	$10^{24} + 1$	17	91	19	$\frac{19}{91}(10^{24} - 1)$
389									$\frac{19}{91}(10^{24} - 1) + (10^{24} - 1)$

Table 5: Examples for Theorem 4

Theorem 5. *Let $a, n \in \mathbb{N}$, with $1 \leq a < n$ and $\gcd(10a, n) = 1$. Suppose that $\operatorname{ord}_n 10 = 2\ell$. Define*

$$g = \gcd\left(n, 10^\ell + 1\right), \quad d = \frac{n}{g}.$$

Let $\lambda \in \{1, \ldots, 9\}$, and let $\mu = \gcd(\lambda, 9)$. Then

(i)

$$S(a, n; \ell, 2) = \lambda_\ell \Longleftrightarrow n = \frac{9}{\mu} g \text{ and } a \equiv \frac{\lambda}{\mu} g \cdot 2^{-1} \pmod{d}, a < \frac{\lambda}{\mu} g.$$

In particular,

$$S(a, n; \ell, 2) = 9_\ell \Longleftrightarrow n \mid (10^\ell + 1) \text{ and } a \in \{1, \ldots, n-1\}, \gcd(a, n) = 1.$$

(ii)

$$S(a, n; \ell, 2) = \lambda_\ell + (10^\ell - 1) \Longleftrightarrow n = \frac{9}{\mu} g \text{ and } a \equiv \frac{\lambda}{\mu} g \cdot 2^{-1} \pmod{d}, a > \frac{\lambda}{\mu} g.$$

Moreover, $S(a, n; \ell, 2) \neq 9_\ell + (10^\ell - 1)$.

Proof. We use the notations of Theorem 4.

(i) $S(a, n; \ell, 2) = \lambda_\ell$ is equivalent to $9k = \lambda d$, and applies only in the case $a < kg$. Note that n is odd and $\gcd(a, n) = 1$ imply $\gcd(2a, d) = 1$. From the proof of Theorem 4, we have $\gcd(g, d) = 1$. Hence, $\gcd(k, d) = 1$, and $\frac{9}{\mu} k = \frac{\lambda}{\mu} d$

implies $d \mid \frac{9}{\mu}$ (because $\gcd(k,d) = 1$) and $\frac{9}{\mu} \mid d$ (because $\gcd\left(\frac{9}{\mu}, \frac{\lambda}{\mu}\right) = 1$). Thus, $d = \frac{9}{\mu}$ and $k = \frac{\lambda}{\mu}$, and so $n = dg = \frac{9}{\mu}g$ and $a \equiv kg \cdot 2^{-1} = \frac{\lambda}{\mu}g \cdot 2^{-1} \pmod{d}$, with $a < kg = \frac{\lambda}{\mu}g$.

When $\lambda = 9$, $\mu = 9$ and so $n = g$ (which is the same as $n \mid (10^\ell + 1)$) and $d = 1$. Thus, there is no congruence restriction on a, and we only have $a < n$.

(ii) $S(a, n; \ell, 2) = \lambda_\ell + (10^\ell - 1)$ is again equivalent to $9k = \lambda d$, and applies only in the case $a > kg$. The arguments in case (i) apply, and we note that $a > n$ when $\lambda = 9$.

□

Example 6.

$$\begin{aligned} \tfrac{5}{63} &= 0.\overline{079365}, \\ \tfrac{59}{63} &= 0.\overline{936507}, \\ \tfrac{5}{51} &= 0.\overline{0980392156862745}, \\ \tfrac{46}{51} &= 0.\overline{9019607843137254}, \\ \tfrac{13}{77} &= 0.\overline{168831}. \end{aligned}$$

a	n	L	b	ℓ	$N(\ell,b)$	g	d	μ	λ	$S(a,n;\ell,b)$
5	63	6	2	3	10^3+1	7	9	1	4	$\lambda \cdot 1_3$
59								1	3	$(\lambda \cdot 1_3) + (10^3 - 1)$
5	51	16	2	8	10^8+1	17	3	3	6	$\lambda \cdot 1_8$
46								3	3	$(\lambda \cdot 1_8) + (10^8 - 1)$
13	77	6	2	3	10^3+1	77	1	3	9	$\lambda \cdot 1_3$

Table 6: Examples for Theorem 5

Proposition 1. *If m and n are positive integers such that* $\gcd(m,n) = \gcd(mn, 10) = 1$, *then* $\operatorname{ord}_{mn} 10 = \operatorname{lcm}(\operatorname{ord}_m 10, \operatorname{ord}_n 10)$.

Proof. From $\gcd(m,n) = 1$, we have $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$, and hence $U(\mathbb{Z}_{mn}) \cong U(\mathbb{Z}_m) \times U(\mathbb{Z}_n)$ via the mapping $a \mapsto (a \bmod m, a \bmod n)$. When $a = 10$, we get $\operatorname{ord}_{mn} 10 = \operatorname{lcm}(\operatorname{ord}_m 10, \operatorname{ord}_n 10)$. □

Proposition 2. *Let $n \in \mathbb{N}$ such that* $\gcd(n, 10) = 1$ *and* $\operatorname{ord}_n 10 = 2\ell$. *Let $n = \prod_{i=1}^t p_i^{\alpha_i}$ be the prime factorization of n, and let 2^{e_i} be the highest power of 2 dividing the period length ℓ_i of the expansion of $1/p_i^{\alpha_i}$, $1 \le i \le t$. If $e = \max\{e_i : 1 \le i \le t\}$, then*

$$\gcd\left(n, 10^\ell + 1\right) = \prod_{i \in \{1,\ldots,t\}:\, e_i = e} p_i^{\alpha_i}.$$

Proof. Let $g = \gcd(n, 10^\ell + 1)$, and let $d = \frac{n}{g}$, as in Theorem 4. Let $\operatorname{ord}_{p_i^{\alpha_i}} 10 = 2^{e_i} \cdot m_i$, m_i odd, $1 \le i \le t$. Then $2\ell = \operatorname{ord}_n 10 = 2^e \cdot \operatorname{lcm}(m_1, \ldots, m_t)$ by Lemma 1.

If $e_i = e$, then $\operatorname{ord}_{p_i^{\alpha_i}} 10 \nmid \ell$. Hence $10^\ell \not\equiv 1 \pmod{p_i^{\alpha_i}}$. Since $10^{2\ell} \equiv 1 \pmod{p_i^{\alpha_i}}$ and $\gcd(10^\ell - 1, 10^\ell + 1) = 1$, we have $p_i^{\alpha_i} \mid \left(10^\ell + 1\right)$.

If $e_i < e$, then $\operatorname{ord}_{p_i} 10 \mid \operatorname{ord}_{p_i^{\alpha_i}} 10 \mid \ell$. Hence $p_i \nmid \left(10^\ell + 1\right)$, since $\gcd(10^\ell - 1, 10^\ell + 1) = 1$. □

Example 7. $2457 = 3^3 \cdot 7 \cdot 13, \qquad 15827 = 7^2 \cdot 17 \cdot 19^2.$

n	L	b	ℓ	$(p_1^{\alpha_1}, \ell_1, e_1)$	$(p_2^{\alpha_2}, \ell_2, e_2)$	$(p_3^{\alpha_3}, \ell_3, e_3)$	e	$\gcd(n, 10^\ell + 1)$
2457	6	2	3	$(3^3, 3, 0)$	$(7, 6, 1)$	$(13, 6, 1)$	1	91
15827	1008	2	504	$(7^2, 6, 1)$	$(17, 16, 4)$	$(19, 18, 1)$	4	17

Table 7: Examples for Proposition 2

Proposition 3. *Let $a, n \in \mathbb{N}$, with $1 \le a < n$ and $\gcd(10a, n) = 1$. Suppose that $\operatorname{ord}_n 10 = 2\ell$. Then every pair of prime divisors of n is period-compatible if and only if $n \mid (10^\ell + 1)$.*

Proof. Let $n = \prod_{i=1}^t p_i^{\alpha_i}$ be the prime factorization of n, and let $\operatorname{ord}_{p_i^{\alpha_i}} 10 = 2^{e_i} \cdot m_i$, m_i odd, $1 \le i \le t$, and let $e = \max\{e_i : 1 \le i \le t\}$, as in Proposition 2. From Proposition 2 we have

$$n \mid (10^\ell + 1) \Longleftrightarrow n = \gcd\left(n, 10^\ell + 1\right) \Longleftrightarrow n = \prod_{e_i = e} p_i^{\alpha_i}. \tag{14}$$

Therefore we must show that every pair of prime divisors of n is period-compatible if and only if $n = \prod_{e_i = e} p_i^{\alpha_i}$.

Let $\operatorname{ord}_{p_i} 10 = 2^{f_i} \cdot n_i$, n_i odd, $1 \le i \le t$. Note that $f_i \le e_i$ for each i. We show that $e_i = e$ implies $f_i = e$. Suppose, to the contrary, that $f_i < e_i = e$ for some i. Recall that $2\ell = \operatorname{ord}_n 10 = 2^e \cdot \operatorname{lcm}(m_1, \ldots, m_t)$ by Lemma 1. Hence $\operatorname{ord}_{p_i} 10$ divides ℓ, and so $p_i \mid (10^\ell - 1)$. Assuming $n \mid (10^\ell + 1)$, we have the contradiction $p_i \mid (10^\ell \pm 1)$. This contradiction shows $e_i = e$ implies $f_i = e$.

We have shown that if $n \mid (10^\ell + 1)$, then by Equation (14) $e_i = e$ for each i, and consequently, $f_i = e$ for each i. Therefore every pair of prime divisors of n is period-compatible.

Conversely, if $n \nmid (10^\ell + 1)$, then again by Equation (14) $e_i < e$ for some i. Thus $f_i < e$ for this i. Let j be such that $e_j = e$. Then $p_j^{\alpha_j} \nmid (10^\ell - 1)$, and hence $p_j^{\alpha_j} \mid (10^\ell + 1)$ since $n \mid (10^\ell - 1)(10^\ell + 1)$ and $\gcd(10^\ell - 1, 10^\ell + 1) = 1$. In particular, $p_j \mid (10^\ell + 1)$. Now if $f_j < e$, then $p_j \mid (10^\ell - 1)$, which together with $p_j \mid (10^\ell + 1)$ is impossible. This contradiction proves $f_j = e$. Hence there exist i, j for which $f_i \ne f_j$, so that p_i, p_j are not period-compatible. □

Remark 2. Equation (14) shows that every pair of distinct maximum prime power divisors of n is period-compatible if and only if $n \mid (10^\ell + 1)$.

Example 8. $77077 = 7^2 \cdot 11^2 \cdot 13$.

n	L	b	ℓ	$(p_1^{\alpha_1}, \ell_1, e_1)$	$(p_2^{\alpha_2}, \ell_2, e_2)$	$(p_3^{\alpha_3}, \ell_3, e_3)$	e	$\gcd(n, 10^\ell + 1)$
77077	462	2	231	$(7^2, 42, 1)$	$(11^2, 22, 1)$	$(13, 6, 1)$	1	77077

Table 8: Examples for Proposition 3

Theorem 6 ([8])**.** *Let $n \in \mathbb{N}$, with $\gcd(n, 10) = 1$. Suppose the decimal expansion of $1/n$ is partitioned into b blocks, each of length ℓ, where $b > 1$.*

(i) *If $\gcd(n, 10^\ell - 1) = 1$, then $(10^\ell - 1)$ divides $S(1, n; \ell, b)$.*

(ii) *If for each prime divisor p of n, ℓ is not a multiple of the period length of $1/p$, then $(10^\ell - 1)$ divides $S(1, n; \ell, b)$.*

(iii) *$S(1, n; \ell, 2) = 10^\ell - 1$ if and only if every pair of prime divisors of n are period-compatible.*

Proof. The results in Theorem 6 follow from our results, as we next show. We use notation from Theorem 1 for parts (i) and (ii), and Theorem 4 and Proposition 3 for part (iii).

If $\gcd(n, 10^\ell - 1) = 1$, then $n \mid N(\ell, b)$ since $n \mid (10^L - 1)$. Therefore $g = n$, $d = 1$, $k = 0$, so that $(10^\ell - 1)$ divides $S(1, n; \ell, b)$. This proves part (i).

Since ℓ is not a multiple of the period length of $1/p$, we have $p \nmid (10^\ell - 1)$. This is true for each prime divisor p of n. Hence $\gcd(n, 10^\ell - 1) = 1$, and the conclusion in part (ii) follows from part (i).

By Proposition 3, every pair of prime divisors of n are period-compatible if and only if $n \mid (10^\ell + 1)$. By Theorem 4, $n \mid (10^\ell + 1)$ if and only if $d = 1$. Now $d = 1$ implies $k = 0$. If $k = 0$ and $a = 1$, then $d \mid 2g^{-1}$ by Theorem 4. But $\gcd(d, g) = 1$ (as shown in the proof of Theorem 4) and d is odd together imply $d = 1$. Thus, $d = 1$ if and only if $k = 0$. We have shown that every pair of prime divisors of n are period-compatible if and only if $k = 0$ (and $d = 1$), which is equivalent to $S(1, n; \ell, 2) = 10^\ell - 1$ again by Theorem 4. This proves part (iii). This is also a direct consequence of Proposition 3 and Corollary 5. ☐

Example 9. $91 = 7 \cdot 13, \qquad 511 = 7 \cdot 73$.

n	L	b	ℓ	$\gcd(n, 10^\ell - 1)$	$S(1, n; \ell, b)$
91	6	2	3	1	$10^3 - 1$
511	24	3	8	73	not a multiple of $10^8 - 1$

Table 9: Examples for Theorem 6, part (i)

$$3131 = 31 \cdot 101$$

n	L	b	ℓ	(p_1, ℓ_1)	(p_2, ℓ_2)	$S(1, n; \ell, b)$
3131	60	6	10	$(31, 15)$	$(101, 4)$	$2(10^{10} - 1)$

Table 10: Examples for Theorem 6, part (ii)

$$1463 = 7 \cdot 11 \cdot 19$$

n	L	b	ℓ	$(p_1^{\alpha_1}, \ell_1, e_1)$	$(p_2^{\alpha_2}, \ell_2, e_2)$	$(p_3^{\alpha_3}, \ell_3, e_3)$	e	$S(1, n; \ell, 2)$
1463	18	2	9	$(7, 6, 1)$	$(11, 2, 1)$	$(13, 6, 1)$	1	$10^9 - 1$

Table 11: Examples for Theorem 6, part (iii)

Acknowledgement. The authors thank the referee for carefully reading this manuscript and suggesting several useful changes.

References

[1] L. E. Dickson, *History of the Theory of Numbers*, Vol. 1, Chelsea Publishing Company, New York, 1952.

[2] B. D. Ginsberg, Midy's (nearly) secret theorem - an extension after 165 years, *College Math. J.* **35** (2004), 26-30.

[3] A. Gupta and B. Sury, Decimal expansion of $1/p$ and subgroup sums, *Integers* **5** (2005), #A19.

[4] R. Honsberger, *Ingenuity in Mathematics*, Random House/Singer, New Mathematical Library, 1970.

[5] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fifth Corrected Edition, Oxford University Press, 1988.

[6] W. G. Leavitt, A theorem on repeating decimals, *Amer. Math. Monthly* **74** (1967), 669-673.

[7] J. Lewittes, Midy's theorem for periodic decimals, *Integers* **7** (2007), #A02.

[8] H. W. Martin, Generalizations of Midy's theorem on repeating decimals, *Integers* **7** (2007), #A03.

[9] E. Midy, De quelques propriétés des nombres et des fractions décimales périodiques, Nantes, 1836.

[10] M. Shrader-Frechette, Complementary rational numbers, *Math. Mag.* **51** (1978), 90-98.

[11] H. Rademacher and O. Toeplitz, *The Enjoyment of Mathematics*, Princeton University Press, Princeton, 1957.