ON THE TWO-COLOR RADO NUMBER FOR $x_1 + ax_2 - x_3 = c$

Srashti Dwivedi
Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi, India
maz138162@maths.iitd.ac.in

Amitabha Tripathi[1]
Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi, India
atripath@maths.iitd.ac.in

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Abstract

Let $a, c \in \mathbb{Z}$, $a \geq 1$. The *Rado number* for the non-homogeneous equation $x_1 + ax_2 - x_3 = c$ in 2 colors is the least positive integer N such that any 2-coloring of the integers in the interval $[1, N]$ admits a monochromatic solution to the given equation. We determine exact values whenever possible, and upper and lower bounds otherwise, for the Rado numbers for all values of c.

1. Introduction

The 2-color *Rado number* for the equation $\mathcal{E}$, denoted by $Rad_2(\mathcal{E})$, is the least positive integer N such that any 2-coloring of the integers in the interval $[1, N]$ admits a monochromatic solution to $\mathcal{E}$. Kosek & Schaal [1] considered the 2-color Rado number for the equation $x_1 + \cdots + x_{m-1} + c = x_m$ for negative values of c. Schaal & Zinter [3] considered the 2-color Rado number for the equation $x_1 + 3x_2 + c = x_3$ for $c \geq -3$. They show that

$$6c + 19 \leq Rad_2\big(x_1 + 3x_2 + c = x_3\big) \leq \begin{cases} 6c + 19 & \text{if } c \equiv k \pmod{k+3},\ k \in S; \\ \frac{13c+41}{2} & \text{if } c \not\equiv k \pmod{k+3},\ k \in S,\ c \text{ is odd}; \\ 7c + 22 & \text{if } c \not\equiv k \pmod{k+3},\ k \in S,\ c \text{ is even}, \end{cases}$$

where $S = \{0, 1, 2, 4, 8, 10, 14, 16, 20\}$.

In this paper, we study the equation $\mathcal{E}_{a,c} : x_1 + ax_2 + c = x_3$ when a is a positive integer and c any integer. We give a necessary and sufficient condition for the Rado number $Rad_2(1, a, -1; c)$ to exist, give upper and lower bounds in all cases, and exact values in many cases. In particular, we determine the Rado number for the

[1]Corresponding author

equation $x_1 + 3x_2 + c = x_3$ for $c \geq -3$. Existence of Rado number requires $\alpha \leq \gamma$, where α and γ are the highest powers of 2 dividing a and c, respectively. The results of this paper are summarized in Table 1.

Conditions on c and a	$\mathbf{Rad}_2(1, a, -1; c)$	**Result**
$\alpha \leq \gamma$ $c < a$, a even $c < a$, a odd $c > a$ $\alpha > \gamma$	exists $\leq \left(a + \frac{a}{2^\alpha} + 2\right)(a - c) + 1$ $\leq (2a+1)(a-c)+1$ $\leq \left(a + \frac{a}{2^\alpha}\right)(c - a) + 1$ does not exist	Theorem 1
$c = a$	1	Proposition 1
$c < a$	$\geq (a+3)(a-c)+1$	Theorem 2
$c \leq -\frac{a(a-3)}{2}$, a odd	$(a+3)(a-c)+1$	Theorem 3
$c \leq 0$, $a \mid c$	$(a+3)(a-c)+1$	Theorem 4
$a \mid c$, $1 < \frac{c}{a} \leq a+1$	$\frac{c}{a}$	Theorem 5
$c > a$	$\geq \left\lceil \frac{1+c(a+3)}{1+a(a+3)} \right\rceil$	Theorem 6
$c \begin{cases} \geq \frac{a(a+2K+1)}{2}, & a \text{ odd} \\ = ma, m \geq a+2, & a \text{ even} \end{cases}$	$\left\lceil \frac{1+c(a+3)}{1+a(a+3)} \right\rceil = K+1$	Theorem 7
$c = \lambda a - \mu$ $\left(\lambda \in [3, a+1],\ \mu \in [1, a+1-\lambda]\right)$	$\geq \lambda + \mu$	Theorem 8

Table 1. Summary of results on the 2-color Rado number for the equation $x_1 + ax_2 - x_3 = c$, $a \geq 1$.

2. Main Results

We study the Rado numbers for the equation

$$x_1 + ax_2 - x_3 = c \tag{1}$$

where a is a positive integer and c is any integer. Throughout this paper, we let $2^\alpha \mid\mid a$ and $2^\gamma \mid\mid c$.

By assigning the color of x_i in the solution of Equation (1) to $x_i - 1$, we note that this is equivalent to determining the smallest positive integer R for which every 2-coloring of $[0, R-1]$ contains a monochromatic solution to

$$x_1 + ax_2 - x_3 = c', \tag{2}$$

where $c' = c - a$.

Theorem 1. *Let $a, c \in \mathbb{Z}$, $a \geq 1$, and let $2^\alpha \mid\mid a$ and $2^\gamma \mid\mid c$. Then $Rad_2(1, a, -1; c)$ exists if and only if $\alpha \leq \gamma$. Moreover, when $\alpha \leq \gamma$, we have*

$$Rad_2(1, a, -1; c) \leq \begin{cases} \left(a + \frac{a}{2^\alpha} + 2\right)(a - c) + 1 & \text{if } c < a,\ a \text{ even}; \\ (2a+1)(a-c)+1 & \text{if } c < a,\ a \text{ odd}; \\ \left(a + \frac{a}{2^\alpha}\right)(c - a) + 1 & \text{if } c > a. \end{cases}$$

Proof. Let $\alpha > \gamma$. Let $\Delta : \mathbb{N} \to \{0,1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } 1 \le x \bmod 2^{\gamma+1} \le 2^{\gamma}; \\ 1 & \text{if } 2^{\gamma} < x \bmod 2^{\gamma+1} \le 2^{\gamma+1}. \end{cases}$$

Reducing Equation (1) modulo $2^{\gamma+1}$ gives $x_1 - x_3 \equiv 2^{\gamma} \pmod{2^{\gamma+1}}$. However, $\Delta(x_1) \neq \Delta(x_3)$, thereby proving that Δ is a valid coloring of $\mathbb{N}$. Therefore, $Rad_2(1, a, -1; c)$ does not exist.

Let $\alpha \le \gamma$. We consider the two cases: (i) $a > c$, and (ii) $a < c$. Write $a = 2^{\alpha} \cdot a_1$ and $c = 2^{\gamma} \cdot c_1$, where a_1, c_1 are both odd. Then $a - c = 2^{\alpha} \cdot t$, where $t = a_1 - 2^{\gamma-\alpha} \cdot c_1 \in \mathbb{Z}$. For the rest of this proof, we consider 2-colorings of $[0, R-1]$ which contain a monochromatic solution to the modified Equation (2).

`Case (a).` Suppose $a > c$. Let $\chi : [0, \left(a + \frac{a}{2^{\alpha}} + 2\right)(a-c)] \to \{0,1\}$ be any 2-coloring of $[0, \left(a + \frac{a}{2^{\alpha}} + 2\right)(a-c)]$. Without loss of generality, let $\chi(0) = 0$. We claim that this forces

$$\chi\big(k(a-c)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a_1 + 2]$.

We use induction on k. Since $x_1 = x_2 = 0$, $x_3 = a-c$ is a solution to Equation (2), we must have $\chi(a-c) = 1$ in order to avoid a monochromatic solution.

Suppose $\chi\big(k(a-c)\big) \equiv k \bmod 2$ for $k \in \{0, 1, 2, \ldots, K-1\}$, $K \le a_1 + 2$.

When K is odd, since $x_1 = (K-1)(a-c)$, $x_2 = 0$, $x_3 = K(a-c)$ is a solution to Equation (2), we must have $\chi\big(K(a-c)\big) = 1$ in order to avoid a monochromatic solution.

Let K be even. In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_1 = (K-1)(a-c)$ and $x_2 = a-c$ implies $\chi\big((a+K)(a-c)\big) = 0$.
- $x_1 = (a+K)(a-c)$ and $x_2 = 0$ implies $\chi\big((a+K+1)(a-c)\big) = 1$.
- $x_2 = a-c$ and $x_3 = (a+K+1)(a-c)$ implies $\chi\big(K(a-c)\big) = 0$.

Note that $t > 0$ in this case. We next claim that $\chi(t) = 0$. Indeed, $x_1 = a-c$, $x_2 = t$, $x_3 = (a_1+2)(a-c)$ forms a monochromatic triple if $\chi(t) = 1$. Finally, $x_1 = 0$, $x_2 = t$, $x_3 = (a_1+1)(a-c)$ forms a monochromatic triple.

We have shown that $\chi\big((a_1+1)(a-c)\big) = 0$ and $\chi\big((a_1+2)(a-c)\big) = 1$. To deduce the color of these two integers, we require $\chi\big((a+a_1+2)(a-c)\big) = 1$, as shown in the argument above. Therefore, any 2-coloring of $[0, \left(a + \frac{a}{2^{\alpha}} + 2\right)(a-c)]$ must admit a monochromatic solution of Equation (2).

When a is odd, note that $a_1 = a$. In this case, we show that any $\chi : [0, (2a+1)(a-c)] \to \{0,1\}$ admits a monochromatic solution of Equation (2). As in the general case, we may assume without loss of generality, that $\chi(0) = 0$. The argument given above shows that

$$\chi\big(k(a-c)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a]$.

In particular, $\chi\big(a(a-c)\big) = 1$. Since $x_1 = a(a-c)$, $x_2 = a-c$, $x_3 = (2a+1)(a-c)$ is a solution to Equation (2), we must have $\chi\big((2a+1)(a-c)\big) = 0$ in order to avoid a monochromatic solution. But then $x_1 = 0$, $x_2 = 2(a-c)$, $x_3 = (2a+1)(a-c)$ forms a monochromatic triple. Therefore, any 2-coloring of $[0, (2a+1)(a-c)]$ must admit a monochromatic solution of Equation (2).

`Case (b).` Suppose $a < c$. We make slight modifications in the argument in Case (a). Let $\chi : [0, \left(a + \frac{a}{2^\alpha}\right)(c-a)] \to \{0, 1\}$ be any 2-coloring of $[0, \left(a + \frac{a}{2^\alpha}\right)(c-a)]$. Without loss of generality, let $\chi(0) = 0$. We claim that this forces

$$\chi\big(k(c-a)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a_1 + 2]$.

We use induction on k.

Since $x_1 = c-a$, $x_2 = x_3 = 0$ is a solution to Equation (2), we must have $\chi(c-a) = 1$ in order to avoid a monochromatic solution.

Suppose $\chi\big(k(c-a)\big) \equiv k \bmod 2$ for $k \in \{0, 1, 2, \ldots, K-1\}$, $K \le a_1 + 2$.

When K is odd, since $x_1 = K(c-a)$, $x_2 = 0$, $x_3 = (K-1)(c-a)$ is a solution to Equation (2), we must have $\chi\big(K(c-a)\big) = 1$ in order to avoid a monochromatic solution.

Let K be even. In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_1 = (K-1)(c-a)$ and $x_2 = c-a$ implies $\chi\big((a+K-2)(c-a)\big) = 0$.
- $x_2 = 0$ and $x_3 = (a+K-2)(c-a)$ implies $\chi\big((a+K-1)(c-a)\big) = 1$.
- $x_2 = c-a$ and $x_3 = (a+K-1)(c-a)$ implies $\chi\big(K(c-a)\big) = 0$.

Note that $t < 0$ in this case. We next claim that $\chi(-t) = 1$. Indeed, $x_1 = 2(c-a)$, $x_2 = -t$, $x_3 = (a_1+1)(c-a)$ forms a monochromatic triple if $\chi(-t) = 0$. Finally, $x_1 = c-a$, $x_2 = -t$, $x_3 = a_1(c-a)$ forms a monochromatic triple.

We have shown that $\chi\big((a_1+1)(c-a)\big) = 0$ and $\chi\big((a_1+2)(c-a)\big) = 1$. To deduce the color of these two integers, we require $\chi\big((a+a_1)(c-a)\big) = 1$, as shown in the argument above. Therefore, any 2-coloring of $[0, \left(a + \frac{a}{2^\alpha}\right)(c-a)]$ must admit a monochromatic solution of Equation (2). □

Proposition 1. *For $a \in \mathbb{N}$, $Rad_2(1, a, -1; a) = 1$.*

Proof. This follows immediately from the fact that $x_1 = x_2 = x_3 = 1$ is a solution to Equation (1). □

3. The Case $c < a$

Theorem 2. *Let $a \geq 1$ and $c < a$. If $\alpha \leq \gamma$, then*

$$Rad_2(1, a, -1; c) \geq (a+3)(a-c)+1.$$

Proof. Let $\Delta : [1, (a+3)(a-c)] \to \{0, 1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in [1, a-c] \cup [(a+2)(a-c)+1, (a+3)(a-c)]; \\ 1 & \text{if } x \in [a-c+1, (a+2)(a-c)]. \end{cases}$$

Suppose x_1, x_2, x_3 is a solution to Equation (1) with $\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$.

Suppose $\Delta(x_i) = 0$ for $i \in \{1, 2, 3\}$. If $x_2 \in [(a+2)(a-c)+1, (a+3)(a-c)]$, then

$$x_3 = x_1 + ax_2 - c \geq 1 + a\big((a+2)(a-c)+1\big) - c > (a+3)(a-c).$$

Hence $x_2 \in [1, a-c]$.

If $x_1 \in [1, a-c]$, then

$$a - c + 1 \leq x_1 + ax_2 - c \leq (a+1)(a-c) - c < (a+2)(a-c)+1.$$

If $x_1 \in [(a+2)(a-c)+1, (a+3)(a-c)]$, then

$$x_3 = x_1 + ax_2 - c \geq (a+2)(a-c)+1+a-c > (a+3)(a-c).$$

Therefore $\Delta(x_i) = 1$ for $i \in \{1, 2, 3\}$, and so

$$x_3 = x_1 + ax_2 - c \geq (a+1)(a-c+1) - c > (a+2)(a-c).$$

This proves that Δ is a valid coloring of $[1, (a+3)(a-c)]$, so that $Rad_2(1, a, -1; c) \geq (a+3)(a-c)+1$. □

Theorem 3. *Let a be odd, $a \geq 1$. If $c \leq -a(a-3)/2$ and $\alpha \leq \gamma$, then*

$$Rad_2(1, a, -1; c) = (a+3)(a-c)+1.$$

Proof. By Theorem 2, it is enough to show that $Rad_2(1, a, -1; c) \leq (a+3)(a-c)+1$. Let $\chi : [0, (a+3)(a-c)] \to \{0, 1\}$ be any 2-coloring of $[0, (a+3)(a-c)]$. Without loss of generality, let $\chi(0) = 0$.

In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_1 = x_2 = 0$ implies $\chi(a-c) = 1$.
- $x_1 = x_2 = a-c$ implies $\chi\big((a+2)(a-c)\big) = 0$.
- $x_2 = 0$ and $x_3 = (a+2)(a-c)$ implies $\chi\big((a+1)(a-c)\big) = 1$.
- $x_1 = (a+2)(a-c)$ and $x_2 = 0$ implies $\chi\big((a+3)(a-c)\big) = 1$.
- $x_2 = a-c$ and $x_3 = (a+3)(a-c)$ implies $\chi\big(2(a-c)\big) = 0$.

- $x_1 = 2(a-c)$ and $x_2 = 0$ implies $\chi\big(3(a-c)\big) = 1$.

We capture this information in the table below.

0	**1**
0	$a-c$
$2(a-c)$	$3(a-c)$
$(a+2)(a-c)$	$(a+1)(a-c)$
	$(a+3)(a-c)$

Table 2. Some initial colorings

We divide the proof into two cases: (i) $\chi(0) = \chi(1)$, and (ii) $\chi(0) \neq \chi(1)$.

`Case (i).` ($\chi(0) = \chi(1)$). We claim that $\chi(n) = 0$ for $1 < n \leq a-c-1$. To do this, we show that $\chi(n) = 0$ for $1 < n \leq a-1$ and that $\chi(m) = \chi(n)$ if $m \equiv n \pmod{a}$ and $1 < m, n \leq a-c-1$.

Assume, by way of contradiction, that $\chi(n) = 1$ for some $n \in \{2, \ldots, a-1\}$. We claim that this forces

$$\chi\big(k(a-c)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a]$.

We use induction on k. The base cases, $k \in \{0, 1, 2, 3\}$, are covered by the arguments in the above paragraph; see Table 2. Suppose $\chi\big(k(a-c)\big) \equiv k \bmod 2$ for $k \in \{0, 1, 2, \ldots, K\}$ for some odd $K < a$.

Let k be odd, $k > 1$. In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_2 = n$ and $x_3 = k(a-c)$ implies $\chi\big(-an + (k-1)(a-c)\big) = 0$. Note that we require $-an+(k-1)(a-c) \geq 0$; it is sufficient to assume $-an+2(a-c) \geq 0$.
- $x_1 = -an + (k-1)(a-c)$ and $x_2 = 0$ implies $\chi\big(-an + k(a-c)\big) = 1$.
- $x_1 = -an + k(a-c)$ and $x_2 = n$ implies $\chi\big((k+1)(a-c)\big) = 0$.
- $x_1 = (k+1)(a-c)$ and $x_2 = 0$ implies $\chi\big((k+2)(a-c)\big) = 1$.

Therefore, for odd k, $\chi\big(k(a-c)\big) = 1$ implies $\chi\big((k+1)(a-c)\big) = 0$ and $\chi\big((k+2)(a-c)\big) = 1$. Since $\chi(a-c) = 1$ (refer Table 2), the proof of our claim is complete.

Thus $\chi\big(a(a-c)\big) = 1$, and the argument in the above paragraph shows that $\chi\big((a+1)(a-c)\big) = 0$, contradicting the results in Table 2. This shows that $\chi(n) = 0$ for $1 < n \leq a-1$.

We next show that $\chi(m) = \chi(n)$ if $m \equiv n \pmod{a}$ and $1 < m, n \leq a-c-1$. In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_1 = 1$ and $x_2 = 0$ implies $\chi\big(1 + (a-c)\big) = 1$.
- $x_1 = n$ and $x_2 = 0$ implies $\chi\big(n + (a-c)\big) = 1$.

- $x_1 = n + (a - c)$ and $x_2 = 1 + (a - c)$ implies $\chi\big(n + a + (a + 2)(a - c)\big) = 0$.
- $x_2 = 0$ and $x_3 = n + a + (a + 2)(a - c)$ implies $\chi\big(n + a + (a + 1)(a - c)\big) = 1$.
- $x_2 = a - c$ and $x_3 = n + a + (a + 1)(a - c)$ implies $\chi(n + a) = 0 = \chi(n)$.

We now have $x_1 = -c$, $x_2 = 1$ and $x_3 = 2(a - c)$ as a monochromatic solution to Equation (2).

`Case (ii).` $(\chi(0) \neq \chi(1))$. We claim that

$$\chi\big(k(a-c)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a]$.

We use induction on k. The base cases, $k \in \{0, 1, 2, 3\}$, are covered by the arguments in the above paragraph; see Table 2. Suppose $\chi\big(k(a-c)\big) \equiv k \bmod 2$ for $k \in \{0, 1, 2, \ldots, K\}$ for some odd $K < a$.

Let k be odd. In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_1 = k(a - c)$ and $x_2 = 1$ implies $\chi\big(a + (k + 1)(a - c)\big) = 0$.
- $x_1 = a + (k + 1)(a - c)$ and $x_2 = 0$ implies $\chi\big(a + (k + 2)(a - c)\big) = 1$.
- $x_2 = 1$ and $x_3 = a + (k + 2)(a - c)$ implies $\chi\big((k + 1)(a - c)\big) = 0$.
- $x_1 = (k + 1)(a - c)$ and $x_2 = 0$ implies $\chi\big((k + 2)(a - c)\big) = 1$.

Therefore, for odd k, $\chi\big(k(a - c)\big) = 1$ implies $\chi\big((k + 1)(a - c)\big) = 0$ and $\chi\big((k + 2)(a-c)\big) = 1$. Since $\chi(a-c) = 1$ (refer Table 2), the proof of our claim is complete.

Thus $\chi\big(a(a - c)\big) = 1$, and the argument in the above paragraph shows that $\chi\big((a + 1)(a - c)\big) = 0$, contradicting the results in Table 2. □

Remark 1. The arguments in Theorem 3 show that the range of c for which the result is valid is in fact more than the statement suggests. In addition to the assumptions made in the theorem, if we write $c \equiv t \pmod{a}$, $0 \leq t \leq a - 1$, then the conclusion of the theorem is valid for

$$c \leq \begin{cases} 0 & \text{if } t \in \{0, a-1\}, \\ -\frac{a(a-t-2)}{2} & \text{if } t \notin \{0, a-1\}. \end{cases}$$

Remark 2. Theorem 3 and Proposition 1 show that $Rad_2(1, 3, -1; c) = 19 - 6c$ for $c \leq 0$, thereby confirming a conjecture of Schaal & Zinter [3], and also for $c = 3$. We can also show that $Rad_2(1, 3, -1; 1) = 14$ and $Rad_2(1, 3, -1; 2) = 8$. We include a proof of these two additional Rado numbers below.

Let $c = 2$. Let $\chi : [1, 8] \to \{0, 1\}$ be any 2-coloring. Suppose, without loss of generality, that $\chi(1) = 0$. In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_1 = x_2 = 1$ implies $\chi(2) = 1$.

- $x_1 = x_2 = 2$ implies $\chi(6) = 0$.
- $x_1 = 6$ and $x_2 = 1$ implies $\chi(7) = 1$.
- $x_2 = 2$ and $x_3 = 7$ implies $\chi(3) = 0$.
- $x_1 = 3$ and $x_2 = 1$ implies $\chi(4) = 1$.
- $x_2 = 1$ and $x_3 = 6$ implies $\chi(5) = 1$.

We capture this information in the table below.

0	**1**
1	2
6	5
3	7
	4

Table 3. Forced colorings for $c = 2$

Table 3 provides a valid 2-coloring of $[1, 7]$. Since both monochromatic pairs $(x_1, x_2) = (1, 3)$, $(x_1, x_2) = (4, 2)$ give $x_3 = 8$, the Rado number equals 8.

Let $c = 1$. Let $\chi : [1, 14] \to \{0, 1\}$ be any 2-coloring. Suppose, without loss of generality, that $\chi(1) = 0$. In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_1 = x_2 = 1$ implies $\chi(3) = 1$.
- $x_1 = x_2 = 3$ implies $\chi(11) = 0$.
- $x_1 = 11$ and $x_2 = 1$ implies $\chi(13) = 1$.
- $x_2 = 1$ and $x_3 = 11$ implies $\chi(9) = 1$.
- $x_2 = 3$ and $x_3 = 13$ implies $\chi(5) = 0$.
- $x_1 = 5$ and $x_2 = 1$ implies $\chi(7) = 1$.
- $x_1 = x_2 = 2$ and $x_3 = 7$ implies $\chi(2) = 0$.
- $x_1 = 2$ and $x_2 = 1$ implies $\chi(4) = 1$.
- $x_1 = 4$ and $x_2 = 3$ implies $\chi(12) = 0$.
- $x_2 = 2$ and $x_3 = 11$ implies $\chi(6) = 1$.
- $x_2 = 1$ and $x_3 = 12$ implies $\chi(10) = 1$.

Note that $\chi(8)$ can be either 0 or 1. We capture this information in the table below.

0	**1**
1	3
11	9, 13
5	7
2	4
12	10
8	6

Table 4. Forced colorings for $c = 1$

Table 4 provides a valid 2-coloring of $[1, 13]$. Since both monochromatic pairs $(x_1, x_2) = (12, 1)$, $(x_1, x_2) = (6, 3)$ give $x_3 = 14$, the Rado number equals 14.

The case $a \mid c$, $c \leq 0$ is covered by Theorem 3 only for odd a. We extend this to all a in the following theorem.

Theorem 4. *Let $a \geq 1$, $c \leq 0$, and let $a \mid c$. Then*

$$Rad_2(1, a, -1; c) = (a + 3)(a - c) + 1.$$

Proof. The existence of $Rad_2(1, a, -1; c)$ is guaranteed by Theorem 1, which holds since $a \mid c$. By Theorem 2, it is enough to show that $Rad_2(1, a, -1; c) \leq (a+3)(a-c) + 1$. Let $\chi : [0, (a + 3)(a - c)] \to \{0, 1\}$ be any 2-coloring of $[0, (a + 3)(a - c)]$. Without loss of generality, let $\chi(0) = 0$.

In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution. This is identical to the initial argument in Theorem 3, but we repeat it for clarity.

- $x_1 = x_2 = 0$ implies $\chi(a - c) = 1$.
- $x_1 = x_2 = a - c$ implies $\chi\big((a + 2)(a - c)\big) = 0$.
- $x_2 = 0$ and $x_3 = (a + 2)(a - c)$ implies $\chi\big((a + 1)(a - c)\big) = 1$.
- $x_1 = (a + 2)(a - c)$ and $x_2 = 0$ implies $\chi\big((a + 3)(a - c)\big) = 1$.
- $x_2 = a - c$ and $x_3 = (a + 3)(a - c)$ implies $\chi\big(2(a - c)\big) = 0$.
- $x_1 = 2(a - c)$ and $x_2 = 0$ implies $\chi\big(3(a - c)\big) = 1$.

We divide the proof into two cases: (i) $\chi(0) = \chi(1)$, and (ii) $\chi(0) \neq \chi(1)$.

`Case (i).` $(\chi(0) = \chi(1))$. We claim that $\chi(n) = 0$ for $1 < n \leq a - c - 1$. Assume, by way of contradiction, that $\chi(n) = 1$ for some $n \in \{2, \ldots, a - c - 1\}$.

In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_1 = x_2 = n$ implies $\chi\big((a + 1)n + (a - c)\big) = 0$.
- $x_1 = a - c$ and $x_2 = n$ implies $\chi\big(an + 2(a - c)\big) = 0$.
- $x_1 = 0$ and $x_3 = an + 2(a - c)$ implies $\chi\big(n + \frac{a-c}{a}\big) = 1$.
- $x_1 = n$ and $x_2 = n + \frac{a-c}{a}$ implies $\chi\big((a + 1)n + 2(a - c)\big) = 0$.

But now $x_1 = (a+1)n+(a-c)$, $x_2 = 0$, $x_3 = (a+1)n+2(a-c)$ is a monochromatic solution to Equation (2). This proves our claim that $\chi(n) = 0$ for $1 < n \leq a-c-1$.

We now have $x_1 = -c$, $x_2 = 1$, $x_3 = 2(a - c)$ as a monochromatic solution to Equation (2).

`Case (ii).` $(\chi(0) \neq \chi(1))$. In each of the following sequences, the color of one of the x_i's is forced in order to avoid a monochromatic solution.

- $x_1 = x_2 = 1$ implies $\chi\big((a + 1) + (a - c)\big) = 0$.
- $x_1 = a - c$ and $x_2 = 1$ implies $\chi\big(a + 2(a - c)\big) = 0$.

- $x_1 = 0$ and $x_3 = a + 2(a - c)$ implies $\chi\big(1 + \frac{a-c}{a}\big) = 1$.
- $x_1 = 1$ and $x_2 = 1 + \frac{a-c}{a}$ implies $\chi\big((a+1) + 2(a-c)\big) = 0$.

But now $x_1 = (a+1) + (a-c)$, $x_2 = 0$, $x_3 = (a+1) + 2(a-c)$ is a monochromatic solution to Equation (2). □

Conjecture 1. Let $a \geq 1$ and let $c \leq 0$. If $\alpha \leq \gamma$, then

$$Rad_2(1, a, -1; c) = (a+3)(a-c) + 1.$$

4. The Case $c > a$

Theorem 5. *Let a, m be integers such that $1 < m \leq a + 1$. Then*

$$Rad_2(1, a, -1; ma) = m.$$

Proof. The existence of $Rad_2(1, a, -1; ma)$ is guaranteed by Theorem 1. The 2-coloring $\Delta : [1, m-1] \to \{0, 1\}$, defined as $\Delta(x) = 0$ for all $x \in [1, m-1]$, is a valid coloring, since $x_1 + ax_2 - x_3 \leq (a+1)(m-1) - 1 = ma - (a - m + 2) < ma$. Hence $Rad_2(1, a, -1; ma) \geq m$.

On the other hand, since $x_1 = x_2 = x_3 = m$ satisfies Equation (1) for $c = ma$, every 2-coloring $\chi : [1, m] \to \{0, 1\}$ admits a monochromatic soloution to Equation (1). Hence $Rad_2(1, a, -1; ma) \leq m$. □

Theorem 6. *Let $a \geq 1$ and $c > a$. If $\alpha \leq \gamma$, then*

$$Rad_2(1, a, -1; c) \geq \left\lceil \frac{1 + c(a+3)}{1 + a(a+3)} \right\rceil.$$

Proof. For convenience, we set

$$K = \left\lceil \frac{1 + c(a+3)}{1 + a(a+3)} \right\rceil - 1,$$

and show that

$$K(a+1)^2 - c(a+2) < K(a+1) - c < K. \tag{3}$$

Both inequalities are equivalent to

$$K < \frac{c}{a}.$$

From the definition of K, since $c > a$, we have

$$K < \frac{1 + c(a+3)}{1 + a(a+3)} < \frac{c}{a}. \tag{4}$$

Thus both inequalities in Equation (3) hold.

Now

$$K(a+1)-c \geq (a+1)\left(\frac{1+c(a+3)}{1+a(a+3)}-1\right)-c = \frac{c(a+2)-a(a+1)(a+3)}{1+a(a+3)} > 0$$

if $c > a(a+2)$.

Let $\Delta : [1,K] \to \{0,1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in \big(\max\{0, K(a+1)^2 - c(a+2)\}, K(a+1)-c\big]; \\ 1 & \text{otherwise.} \end{cases}$$

We claim that Δ provides a valid 2-coloring of $[1,K]$ with respect to Equation (1).

If $\Delta(x_1) = \Delta(x_2) = 0$, then

$$x_3 \leq (a+1)\big(K(a+1)-c\big) - c = K(a+1)^2 - c(a+2).$$

Hence $\Delta(x_3) = 1$. Therefore, we must have $\Delta(x_i) = 1$ for $i \in \{1,2,3\}$.

If $x_i > K(a+1) - c$ for $i \in \{1,2\}$, then

$$K(a+1)^2 - c(a+2) = (a+1)\big(K(a+1)-c\big) - c < x_1 + ax_2 - c = x_3 \leq K(a+1) - c.$$

Therefore, $x_i \in \big[1, K(a+1)^2 - c(a+2)\big]$ for at least one $i \in \{1,2\}$. Now

$$x_3 \leq K(a+1)^2 - c(a+2) + Ka - c = K\big(1 + a(a+3)\big) - c(a+3) \leq 0,$$

by Equation (4), so that x_3 is outside the domain of Δ.

We have shown that $K(a+1)^2 - c(a+2) < K(a+1) - c$ for $c > a$, and further that $K(a+1) - c > 0$ if $c > a(a+2)$. Thus, Δ provides a valid 2-coloring for $c > a(a+2)$. For $c \in [a+1, (a+1)^2)$, it may be the case that $K(a+1) - c < 1$, in which case all integers in the interval $[1,K]$ are colored 1. Since $x_3 = x_1 + ax_2 - c \leq K(a+1) - c$, Δ provides a valid 1-coloring if $K(a+1) - c < 1$. Therefore $Rad_2(1,a,-1;c) > K$ in any case. □

Theorem 7. *Let $a, c \in \mathbb{N}$, $c > a$, and let $\alpha \leq \gamma$. Then for*

$$c \begin{cases} \geq \frac{a(a+2K+1)}{2} & \textit{if } a \textit{ is odd;} \\ = ma, m \geq a+2 & \textit{if } a \textit{ is even,} \end{cases},$$

where $K = \left\lceil \frac{1+c(a+3)}{1+a(a+3)} \right\rceil - 1$, we have

$$Rad_2(1,a,-1;c) = \left\lceil \frac{1+c(a+3)}{1+a(a+3)} \right\rceil.$$

Proof. Suppose that $c > a(a+2)$. By Theorem 6, it suffices to prove that

$$Rad_2(1,a,-1;c) \leq \left\lceil \frac{1+c(a+3)}{1+a(a+3)} \right\rceil = K+1,$$

where $K = \left\lceil \frac{1+c(a+3)}{1+a(a+3)} \right\rceil - 1$, as defined in Theorem 6.

Also

$$K \geq \frac{(c-a)(a+3)}{1+a(a+3)} > a \tag{5}$$

when $c > a(a+2)$.

Let $\chi : [1, K+1] \to \{0,1\}$ be any 2-coloring of the integers in the interval $[1, K+1]$. Consider the complimentary coloring $\overline{\chi} : [1, K+1] \to \{0,1\}$ given by

$$\overline{\chi}(x) = \chi(K+2-x).$$

Then monochromatic solutions to Equation (1) under χ correspond to monochromatic solutions to

$$x_1 + ax_2 - x_3 = (K+2)a - c \tag{6}$$

under $\overline{\chi}$.

From Equation (4), $(K+2)a - c < a\left(\frac{c}{a}+2\right) - c = 2a$. Thus, we have $c' = (K+2)a - c < 2a$ for $c > a(a+2)$. If $c' \leq -\frac{a(a-3)}{2}$, then every 2-coloring of $[1, (a-c')(a+3)+1]$ admits a monochromatic solution to Equation (6) by Theorem 3 and Theorem 4. Now using the definition of K, we have

$$\begin{aligned} (a-c')(a+3)+1 &= \big(c-(K+1)a\big)(a+3)+1 \\ &= 1 + c(a+3) - \big(1+a(a+3)\big)(K+1) + K + 1 \\ &\leq K+1. \end{aligned}$$

Hence every 2-coloring of $[1, K+1]$ also admits a monochromatic solution to Equation (1) in this case. □

Remark 3. The arguments in Theorem 7 show that the range of c for which the result is valid is in fact more than the statement suggests. In addition to the assumptions made in the theorem, if we write $c \equiv t \pmod{a}$, $0 \leq t \leq a-1$, then the conclusion of the theorem is valid for

$$c \geq \begin{cases} a(K+2) & \text{if } t \in \{0, a-1\}, \\ -\frac{a(a-t+2K+2)}{2} & \text{if } t \notin \{0, a-1\}. \end{cases}$$

Theorem 8. *Let $a \geq 1$, and let $c = \lambda a - \mu$, with $3 \leq \lambda \leq a+1$ and $1 \leq \mu \leq a+1-\lambda$. If $\alpha \leq \gamma$, then*

$$Rad_2(1, a, -1; c) \geq \lambda + \mu.$$

Proof. Suppose $c = \lambda a - \mu$, with $3 \leq \lambda \leq a+1$, $1 \leq \mu \leq a+1-\lambda$. Let $\Delta : [0, \lambda+\mu-2] \to \{0,1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in [0, \lambda-2]; \\ 1 & \text{if } x \in [\lambda-1, \lambda+\mu-2]. \end{cases}$$

We claim that Δ provides a valid 2-coloring of $[0, \lambda + \mu - 2]$ with respect to Equation (2).

Suppose $\Delta(x_1) = \Delta(x_2) = 0$. Then

$$x_3 = x_1 + ax_2 - (\lambda - 1)a + \mu \leq (\lambda - 2)(a + 1) - (\lambda - 1)a + \mu \leq -1,$$

so that x_3 is outside the domain of Δ. Therefore, we must have $\Delta(x_i) = 1$ for $i \in \{1, 2, 3\}$. But then

$$x_3 = x_1 + ax_2 - (\lambda - 1)a + \mu \geq (\lambda - 1)(a + 1) - (\lambda - 1)a + \mu \geq \lambda + \mu - 1,$$

and again x_3 is outside the domain of Δ. Hence $Rad_2(1, a, -1; c) \geq \lambda + \mu$. □

Conjecture 2. Let $a \geq 1$, and $c \geq a(K + 2)$, where $K = \left\lceil \frac{1+c(a+3)}{1+a(a+3)} \right\rceil - 1$. If $\alpha \leq \gamma$, then

$$Rad_2(1, a, -1; c) = K + 1.$$

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