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ON THE TWO-COLOR RADO NUMBER FOR $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c$

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Abstract

Let $a, c, m \in \mathbb{Z}$, $4 \leq m \leq a$. The 2-color Rado number for the equation $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c$ is the least positive integer N such that any 2-coloring of $\{1, \ldots, N\}$ admits a monochromatic solution to the given equation. We determine exact values whenever possible, and upper and lower bounds otherwise, of Rado numbers for all values of c. This generalizes a recent work of the authors concerning the case of $m = 3$.

1. Introduction

The *r-color Rado number* for an equation $\mathscr{E}$, denoted by $\texttt{Rad}_r(\mathscr{E})$, is the least positive integer N such that any r-coloring of $\{1, \ldots, N\}$ admits a monochromatic solution to $\mathscr{E}$. There has been considerable interest in the study of Rado numbers in the past four decades, particularly in the case $r = 2$; see for instance [1–14, 16–18]. An introductory study of Rado numbers and a comprehensive list of references may be found in [15].

Schaal and Zinter [18] considered the 2-color Rado number for the non-homogeneous equation $x_1 + 3x_2 + c = x_3$ for $c \geq -3$, and found exact values in some cases and bounds in others. This was recently generalized to the non-homogeneous equation $\mathscr{E}_{a,c} : x_1 + ax_2 - x_3 = c$ by Dwivedi and Tripathi [5], who gave necessary and sufficient conditions for existence of the Rado number of $\mathscr{E}_{a,c}$, determined exact values whenever possible, and upper and lower bounds otherwise, for all values of a and c. In this paper, we study the non-homogeneous equation $\mathscr{E}_{a,c;m} : \sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c$ when $4 \leq m \leq a$. We give a necessary condition and a sufficient condition for the Rado number to exist, give upper and lower bounds in all cases, and exact values in many cases. Throughout this paper, we denote the Rado number of $\mathscr{E}_{a,c;m}$ by $\mathscr{R}(a, c; m)$.

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We have divided the results of this paper into three sections. Section 2 is about existence of $\mathscr{R}(a,c;m)$. We show that $\mathscr{R}(a,c;m)$ does not exist when $c(a+m)$ is odd (Theorem 1), and give upper bounds for $\mathscr{R}(a,c;m)$ when $a+m$ is even (Theorem 2) and when c is even under some restrictions (Theorem 3). These results imply $\mathscr{R}(a,c;m)$ exists if and only if $c(a+m)$ is even, except possibly when $a+m$ is odd and for some even values of c (Theorem 4). Under the assumption $\mathscr{R}(a,c;m)$ exists, we deal with the cases $c < a+m-3$ and $c > a+m-3$ in Section 3 and Section 4, respectively. In Section 3, we provide a lower bound for $\mathscr{R}(a,c;m)$ when $c < a+m-3$ (Theorem 5), which together with the two upper bounds for $\mathscr{R}(a,c;m)$, provides bounds in this case. We obtain the exact value of $\mathscr{R}(a,c;m)$ when $c < -(a-2)(a+m-3)$ (Theorem 6), and conjecture this value of $\mathscr{R}(a,c;m)$ to hold for all $c \leq 0$ (Conjecture 1). In Section 4, we provide a lower bound for $\mathscr{R}(a,c;m)$ when $c > a+m-3$ (Theorem 8), which together with the two upper bounds for $\mathscr{R}(a,c;m)$, provides bounds in this case. We obtain the exact value of $\mathscr{R}(a,c;m)$ when $a+m-3$ divides c (Theorem 7) and when $c > (a+\kappa)(a+m-3)$ (Theorem 9). For $a+m-3 < c < (a+\kappa)(a+m-3)$, we provide a better lower bound for $\mathscr{R}(a,c;m)$ (Theorem 10) than the one in Theorem 8. The results of this paper are summarized in Table 1.

Conditions on c and a	$\mathscr{R}(a,c;m)$	**Result**
$c(a+m)$ odd	condition for non-existence	Theorem 1
$c < a+m-3$ $c > a+m-3$ (a,m same parity)	$\leq (2a+m-3)(a+m-c-3)+1$ $\leq (2a+m-5)(c-a-m+3)+1$	Theorem 2
$c < a+m-3$ $c > a+m-3$ $\left(2^\alpha \\| a,\ 2^\gamma \\| c,\ 2^\alpha \\| (m-3)\right)$ $(1 \leq \alpha \leq \gamma)$	$\leq \left(a+\frac{a}{2^a}+2m-4\right)(a+m-c-3)+1$ $\leq \left(a+\frac{a}{2^a}+2m-8\right)(c-a-m+3)+1$	Theorem 3
$a+m$ even a,c even, m odd $\left(2^\alpha \\| a,\ 2^\gamma \\| c,\ 2^\alpha \\| (m-3)\right)$ $(1 \leq \alpha \leq \gamma)$	condition for existence	Theorem 4
$c < a+m-3$	$\geq (a+m)(a+m-c-3)+1$	Theorem 5
$c < -(a-2)(a+m-3)$ (a,m same parity)	$(a+m)(a+m-c-3)+1$	Theorem 6
$c = k(a+m-3)$ $(1 \leq k \leq a+m-2)$	k	Theorem 7
$c > a+m-3$	$\geq \left\lceil \frac{1+c(a+m)}{1+(a+m)(a+m-3)} \right\rceil$	Theorem 8
$c > (a+\kappa)(a+m-3)$ (a,m same parity)	$\kappa = \left\lceil \frac{1+c(a+m)}{1+(a+m)(a+m-3)} \right\rceil - 1$	Theorem 9
$c = \lambda(a+m-3)-\mu$ $3 \leq \lambda \leq a+m-2$ $1 \leq \mu \leq a+m-2-\lambda$	$\geq \lambda+\mu$	Theorem 10

Table 1: Summary of results on the 2-color Rado number for the equation $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c$, $4 \leq m \leq a$.

2. Preliminaries and Existential Results

For positive integers N, by $[1, N]$ we mean the set $\{1, 2, 3 \ldots, N\}$. By a 2-*coloring* of $[1, N]$ we mean a mapping $\chi : [1, N] \to \{0, 1\}$. It is customary to allow distinct variables in a given equation $\mathscr{E}$ to take the same values for purposes of computing Rado numbers. A 2-coloring χ of $[1, N]$ is said to *monochromatic* for the equation $\mathscr{E}$ if $\chi(x_i)$ is constant for each variable x_i in $\mathscr{E}$, and *valid* for the equation $\mathscr{E}$ if $\chi(x_i) \neq \chi(x_j)$ for some pair of distinct variables x_i, x_j satisfying $\mathscr{E}$.

To prove that a positive integer U is an upper bound for the Rado number $\mathtt{Rad}_2\big(\mathscr{E}\big)$ we must show that every 2-coloring χ of $[1, N]$ admits a monochromatic solution for the equation $\mathscr{E}$. To prove that a positive integer L is a lower bound for $\mathtt{Rad}_2\big(\mathscr{E}\big)$ we must show the existence of a valid 2-coloring of $[1, L-1]$ for the equation $\mathscr{E}$. Together, these two arguments provide the bounds $L \leq \mathtt{Rad}_2\big(\mathscr{E}\big) \leq U$. The goal of studying Rado numbers is to determine $\mathtt{Rad}_2\big(\mathscr{E}\big)$, and when that is not possible, to find bounds U and L which are as close as possible.

In this paper, we study the 2-color Rado number for the equation

$$\sum_{i=1}^{m-2} x_i + a x_{m-1} - x_m = c \tag{1}$$

where a, m are positive integers and c is any integer. The case $m = 3$ was studied by the present authors in [5]. We assume $4 \leq m \leq a$ throughout this paper, and denote these 2-color Rado numbers by

$$\mathtt{Rad}_2\big(\underbrace{1, \ldots, 1}_{m-2 \text{ times}}, a, -1; c\big),$$

or more briefly by $\mathscr{R}(a, c; m)$, as stated in the Introduction. Limited data that we worked with indicate that our results do not extend to the cases $m > a$.

The computation of the 2-color Rado number involves assigning colors to each integer in $[1, N]$, and in particular, to each x_i in the solution of Equation (1), as long as $x_i \in [1, N]$. In some cases, it is more beneficial to consider the translated 2-coloring of the integers in $[0, N-1]$ by assigning the color of x_i in the solution of Equation (1) to $x_i - 1$. This translation results in solving the equivalent problem of determining the smallest positive integer N for which every 2-coloring of $[0, N-1]$ contains a monochromatic solution to

$$\sum_{i=1}^{m-2} x_i + a x_{m-1} - x_m = c', \tag{2}$$

where $c' = c - a'$ and $a' = a + m - 3$.

We show the non-existence of $\mathscr{R}(a, c; m)$ when $c(a + m)$ is odd by providing a valid 2-coloring of the set of positive integers for the equation $\mathscr{E}_{a,c;m}$. We show the existence of $\mathscr{R}(a, c; m)$ when $a + m$ is even, or when a, c are both even and m is odd under some restrictions, by exhibiting suitable choices of R such that every

2-coloring of $[1, R]$ admits a monochromatic solution to Equation (1). The existence of R not only shows the existence of $\mathscr{R}(a, c; m)$ but also serves as an upper bound for $\mathscr{R}(a, c; m)$ in these cases.

Theorem 1. *Let $a, c, m \in \mathbb{Z}$ and $4 \leq m \leq a$. Then $\mathscr{R}(a, c; m)$ does not exist if $c(a + m)$ is odd.*

Proof. Assume that $a + m$ and c are both odd. Define the coloring $\chi : \mathbb{N} \to \{0, 1\}$ by $\chi(i) \equiv i \pmod 2$. We show that this coloring does not admit a monochromatic solution to Equation (1). By way of contradiction, assume there exists a monochromatic solution $x_1, \ldots, x_m$, which means that $x_1, \ldots, x_m$ are of the same parity. But then $c = \sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m$ has the same parity as $a+m-3$, which contradicts our assumption. □

We first show the existence of $\mathscr{R}(a, c; m)$ when $a + m$ is even.

Theorem 2. *Let $4 \leq m \leq a$, with a, m of the same parity. Then*

$$\mathscr{R}(a, c; m) \leq \begin{cases} (2a + m - 3)(a + m - c - 3) + 1 & \textit{if } c < a + m - 3; \\ (2a + m - 5)(c - a - m + 3) + 1 & \textit{if } c > a + m - 3. \end{cases}$$

Proof. Let $a' = a + m - 3$ throughout this proof. We consider the two cases (i) $c < a'$, (ii) $c > a'$ separately. In each case, we show that every 2-coloring of $[0, R-1]$ exhibits a monochromatic solution to the translated Equation (2).

`Case (i):` $c < a'$. Let $\chi : [0, (a' + a)(a' - c)] \to \{0, 1\}$ be any 2-coloring of $[0, (a' + a)(a' - c)]$. Without loss of generality, let $\chi(0) = 0$. Assume, by way of contradiction, that χ is a valid coloring, which means that there is no monochromatic solution to Equation (2) under χ.

When a, m are both even, we claim that

$$\chi\big(k(a' - c)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a - 1]$.

We use induction on k. With each $x_i = 0$ for $1 \leq i \leq m - 1$ in Equation (2), $x_m = a' - c$, so that $\chi(a' - c) = 1$. Suppose $\chi\big(k(a' - c)\big) \equiv k \pmod 2$ for $k \in \{0, 1, 2, \ldots, K - 1\}$ where $K \leq a - 1$.

Let K be odd. Since $x_1 = (K-1)(a'-c)$, $x_i = 0$ for $2 \leq i \leq m-1$, $x_m = K(a'-c)$ is a solution Equation (2), and $(K - 1)(a' - c)$ and 0 are both colored 0, we must have $\chi\big(K(a' - c)\big) = 1$, as χ is a valid coloring.

Let K be even. Each step in the following sequence forces a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2), so that χ remains a valid coloring.

- $x_1 = (K - 1)(a' - c)$, $x_i = a' - c$ for $2 \leq i \leq m - 1$ imply $\chi\big((a + K + m - 3)(a' - c)\big) = 0$.

- $x_1 = (a+K+m-3)(a'-c)$, $x_i = 0$ for $2 \le i \le m-1$ imply $\chi\big((a+K+m-2)(a'-c)\big) = 1$.
- $x_i = a'-c$ for $2 \le i \le m-1$, $x_m = (a+K+m-2)(a'-c)$ imply $\chi\big(K(a'-c)\big) = 0$.

We have shown that $\chi\big((a-2)(a'-c)\big) = 0$ and $\chi\big((a-1)(a'-c)\big) = 1$. To deduce the color of these two integers, we require $\chi\big((2a+m-4)(a'-c)\big) = 1$, as shown in the argument above. Since $x_1 = (a-1)(a'-c)$, $x_i = a'-c$ for $2 \le i \le m-1$, $x_m = (2a+m-3)(a'-c)$ is a solution to Equation (2), and $a'-c$ and $(a-1)(a'-c)$ are both colored 1, we must have $\chi\big((2a+m-3)(a'-c)\big) = 0$ as χ is a valid coloring. Since $m \le a$ and a, m are both even, we have $\chi\big((m-4)(a'-c)\big) = 0$. But then $x_1 = (m-4)(a'-c)$, each $x_i = 0$ for $2 \le i \le m-2$, and $x_{m-1} = 2(a'-c)$, $x_m = (2a+m-3)(a'-c)$ forms a monochromatic solution to Equation (2). This contradicts our assumption that χ is a valid coloring.

When a, m are both odd, a similar argument shows that

$$\chi\big(k(a'-c)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a-1]$.

In this case, we have $\chi\big((2a+m-3)(a'-c)\big) = 1$ and $\chi\big((a-1)(a'-c)\big) = 0$. Since $x_i = a'-c$ for $1 \le i \le m-1$, $x_m = (a+m-1)(a'-c)$ is a solution to Equation (2), and $\chi(a'-c) = 1$ as χ is a valid coloring. But now we have the monochromatic solution $x_1 = (a-1)(a'-c)$, $x_2 = (m-1)(a'-c)$, $x_i = 0$ for $3 \le i \le m-1$, and $x_m = (a+m-1)(a'-c)$, to Equation (2). This again contradicts our assumption that χ is a valid coloring.

Therefore any 2-coloring of $[0, (a'+a)(a'-c)]$ must admit a monochromatic solution of Equation (2), and so any 2-coloring of $[1, (a'+a)(a'-c)+1]$ must admit a monochromatic solution of Equation (1). Hence, we obtain the upper bound.

`Case (ii):` $c > a'$. We make slight modifications in the argument in Case (i). Let $\chi : [0, (a'+a-2)(c-a')] \to \{0, 1\}$ be any 2-coloring of $[0, (a'+a-2)(c-a')]$. Without loss of generality, let $\chi(0) = 0$, and by way of contradiction, suppose that χ is a valid coloring.

When a, m are both odd, we claim that

$$\chi\big(k(c-a')\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a-1]$.

We use induction on k. With $x_1 = c-a'$ and $x_i = 0$ for $2 \le i \le m-1$ in Equation (2), we have $x_m = 0$, so that $\chi(c-a') = 1$. Suppose $\chi\big(k(c-a')\big) \equiv k \pmod 2$ for $k \in \{0, 1, 2, \ldots, K-1\}$ for $K \le a-1$.

Let K be odd. Since $x_1 = K(c-a')$, $x_i = 0$ for $2 \le i \le m-1$, $x_m = (K-1)(c-a')$ is a solution to Equation (2), and since $K(c-a')$ and 0 are colored 0, we must have $\chi\big(K(c-a')\big) = 1$ as χ is a valid coloring.

Let K be even. Each step in the following sequence forces a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2), so that χ remains a valid coloring.

- $x_1 = (K-1)(c-a')$, $x_i = c - a'$ for $2 \le i \le m-1$ imply $\chi\big((a+K+m-5)(c-a')\big) = 0$.
- $x_i = 0$ for $2 \le i \le m-1$, $x_m = (a+K+m-5)(c-a')$ imply $\chi\big((a+K+m-4)(c-a')\big) = 1$.
- $x_i = c-a'$ for $2 \le i \le m-1$, $x_m = (a+K+m-4)(c-a')$ imply $\chi\big(K(c-a')\big) = 0$.

From the second deduction we have $\chi\big((2a+m-5)(c-a')\big) = 1$, and we can use this to show $\chi\big((a-1)(c-a')\big) = 0$. Since $x_i = c-a'$ for $1 \le i \le m-1$, $x_m = (a+m-3)(c-a')$ is a solution to Equation (2), and since $\chi(c-a') = 1$, we must have $\chi\big((a+m-3)(c-a')\big) = 0$ as χ is a valid coloring. But then $x_1 = (a-1)(c-a')$, $x_2 = (m-1)(c-a')$, $x_i = 0$ for $3 \le i \le m-1$, $x_m = (a+m-3)(c-a')$ is a monochromatic solution to Equation (2). This contradicts our assumption that χ is a valid coloring.

When a, m are both even, a similar argument shows that

$$\chi\big(k(c-a')\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a-1]$.

In this case, we have $\chi\big((a-1)(c-a')\big) = 1$. With $x_1 = (a-1)(c-a')$, $x_i = c-a'$ for $2 \le i \le m-1$ in Equation (2), $x_m = (2a+m-5)(c-a')$, so that $\chi((2a+m-5)(c-a')) = 0$. Now $x_1 = (m-4)(c-a')$, $x_i = 0$ for $2 \le i \le m-2$, $x_{m-1} = 2(c-a')$, $x_m = (2a+m-5)(c-a')$ is a monochromatic solution to Equation (2). This again contradicts our assumption that χ is a valid coloring.

Therefore, any 2-coloring of $[0, (a'+a-2)(c-a')]$ must admit a monochromatic solution of Equation (2), and so any 2-coloring of $[1, (a'+a-2)(c-a')+1]$ must admit a monochromatic solution of Equation (1). Hence, we obtain the upper bound. □

We next show the existence of $\mathscr{R}(a, c; m)$ when both a and c is even and m is odd, but under the conditions imposed by Theorem 3 stated below. By $p^e \,||\, n$ we mean $p^e \mid n$ and $p^{e+1} \nmid n$.

Theorem 3. *Let $4 \le m \le a$, with a even and m odd. If $2^\alpha \,||\, a$, $2^\gamma \,||\, c$, $\alpha \le \gamma$, and $2^\alpha \,||\, (m-3)$, then*

$$\mathscr{R}(a, c; m) \le \begin{cases} \big(a + \frac{a}{2^\alpha} + 2m - 4\big)(a+m-c-3) + 1 & \text{if } c < a+m-3; \\ \big(a + \frac{a}{2^\alpha} + 2m - 8\big)(c-a-m+3) + 1 & \text{if } c > a+m-3. \end{cases}$$

Proof. Let $a' = a + m - 3$ throughout this proof. We consider the two cases: (i) $c < a'$, (ii) $c > a'$ separately. In each case, we show that every 2-coloring of $[0, R-1]$ exhibits a monochromatic solution to the translated Equation (2). Write $a = 2^\alpha \cdot a_1$, $c = 2^\gamma \cdot c_1$, $m-3 = 2^\alpha \cdot m_1$, where a_1, c_1, m_1 are all odd. Then $a' - c = 2^\alpha \cdot t$, where $t = a_1 + m_1 - 2^{\gamma-\alpha} \cdot c_1 \in \mathbb{Z}$.

`Case (i):` $c < a'$. Let $\chi : [0, \big(a + \frac{a}{2^\alpha} + 2m - 4\big)(a' - c)] \to \{0, 1\}$ be any 2-coloring of $[0, \big(a + \frac{a}{2^\alpha} + 2m - 4\big)(a' - c)]$. Without loss of generality, let $\chi(0) = 0$. Assume, by way of contradiction, that χ is a valid coloring. We claim that this implies

$$\chi\big(k(a'-c)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in \big[0, \frac{a}{2^\alpha} + m - 1\big]$.

We use induction on k to show this along the lines of Theorem 2, Case (i). The details are omitted. Note that $t > 0$ in this case. Since $x_i = 0$ for $1 \le i \le m-2$, $x_{m-1} = t$, $x_m = \big(\frac{a}{2^\alpha}+1\big)(a'-c)$ is a solution to Equation (2), and since $(\frac{a}{2^\alpha}+1)(a'-c)$ and 0 are colored the same, we must have $\chi(t) = 1$ since χ is a valid coloring.

We have shown that $\chi\big((\frac{a}{2^a}+m-2)(a'-c)\big) = 0$ and $\chi\big((\frac{a}{2^a}+m-1)(a'-c)\big) = 1$. To deduce the color of these two integers, we require $\chi\big((a+\frac{a}{2^a}+2m-4)(a'-c)\big) = 1$, as shown in the argument above. But then $x_i = a'-c$ for $1 \le i \le m-2$, $x_{m-1} = t$, $x_m = \big(\frac{a}{2^a}+m-1\big)(a'-c)$ is a monochromatic solution to Equation (2). Therefore, any 2-coloring of $\big[0, \big(a+\frac{a}{2^\alpha}+2m-4\big)(a'-c)\big]$ must admit a monochromatic solution of Equation (2), and so any 2-coloring of $\big[1, \big(a+\frac{a}{2^\alpha}+2m-4\big)(a'-c)+1\big]$ must admit a monochromatic solution of Equation (1). Hence, we obtain the upper bound.

`Case (ii):` $c > a'$. We make slight modifications in the argument in Case (i). Let $\chi : \big[0, \big(a + \frac{a}{2^\alpha} + 2m - 8\big)(c - a')\big] \to \{0, 1\}$ be any 2-coloring of $\big[0, \big(a + \frac{a}{2^\alpha} + 2m - 8\big)(c - a')\big]$. Without loss of generality, let $\chi(0) = 0$, and assume by way of contradiction that χ is a valid coloring. An argument similar to Case (i) shows that

$$\chi\big(k(c-a')\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in \big[0, \frac{a}{2^\alpha} + m - 3\big]$. Note that $t < 0$ in this case. Since $x_i = 0$ for $1 \le i \le m-2$, $x_{m-1} = -t$, $x_m = \big(\frac{a}{2^\alpha} - 1)(c - a')$ is a solution to Equation (2), and since $(\frac{a}{2^\alpha} - 1)(c - a')$ and 0 are colored 0, we must have $\chi(-t) = 1$ since χ is a valid coloring.

We have shown that $\chi\big(\big(\frac{a}{2^a}+m-4\big)(c-a')\big) = 0$ and $\chi\big(\big(\frac{a}{2^a}+m-3\big)(c-a')\big) = 1$. To deduce the color of these two integers, we require $\chi\big(\big(a+\frac{a}{2^a}+2m-8\big)(c-a')\big) = 1$, as shown in the argument above. But then $x_i = c-a'$ for $1 \le i \le m-2$, $x_{m-1} = -t$, $x_m = \big(\frac{a}{2^a}+m-3\big)(c-a')$ is a monochromatic solution to Equation (2). Therefore, any 2-coloring of $\big[0, \big(a+\frac{a}{2^\alpha}+2m-8\big)(c-a')\big]$ must admit a monochromatic solution of Equation (2), and so any 2-coloring of $\big[1, \big(a+\frac{a}{2^\alpha}+2m-8\big)(c-a')+1\big]$ must admit a monochromatic solution of Equation (1). Hence, we obtain the upper bound. □

Theorem 1 shows that $\mathscr{R}(a,c;m)$ does not exist if $c(a+m)$ is odd and Theorem 2 shows that $\mathscr{R}(a,c;m)$ exists if $a+m$ is even. Theorem 3 covers some of the cases where $a+m$ is odd and c is even, but not all. Covering the remaining cases would lead to the desirable result that $\mathscr{R}(a,c;m)$ exists if and only if $c(a+m)$ is even.

Theorem 4. *Let $a,c,m \in \mathbb{Z}$ and $4 \leq m \leq a$. Then*

(i) *$\mathscr{R}(a,c;m)$ does not exist if $c(a+m)$ is odd;*

(ii) *$\mathscr{R}(a,c;m)$ exists if $a+m$ is even or if $2^\alpha \mid\mid a$, $2^\gamma \mid\mid c$, $2^\alpha \mid\mid (m-3)$, where $1 \leq \alpha \leq \gamma$.*

Proof. These results are due to Theorem 1, Theorem 2 and Theorem 3. □

3. The Case $c < a + m - 3$

We deal with the case $c < a+m-3$ in this section. Theorem 2 and Theorem 3 provide upper bounds for $\mathscr{R}(a,c;m)$ when $a+m$ is even, and in some cases where c is even and $a+m$ is odd. We provide a lower bound for $\mathscr{R}(a,c;m)$ in this section, so that we have both lower and upper bounds in cases where $\mathscr{R}(a,c;m)$ exists. We further show this lower bound is also an upper bound when $c < -(a-2)(a+m-3)$, thereby achieving the exact value of $\mathscr{R}(a,c;m)$ in these cases. Some computer aided programmes suggest the same formula for $\mathscr{R}(a,c;m)$ extends to all $c \leq 0$, but not to $0 < c < a+m-3$.

Theorem 5. *Let $4 \leq m \leq a$. For $c < a+m-3$, we have*

$$\mathscr{R}(a,c;m) \geq (a+m)(a+m-c-3)+1.$$

Proof. Let $c' = a' - c = a+m-3-c$ throughout this proof. Let $\Delta : [1,(a+m)c'] \to \{0,1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in [1,c'] \cup [(a+m-1)c'+1,(a+m)c']; \\ 1 & \text{if } x \in [c'+1,(a+m-1)c']. \end{cases}$$

Let $A = [1,c']$, $B = [c'+1,(a+m-1)c']$, and $C = [(a+m-1)c'+1,(a+m)c']$. Suppose $x_1,\ldots,x_m$ is a solution to Equation (1) with $\Delta(x_1) = \cdots = \Delta(x_m)$.

Suppose $\Delta(x_i) = 0$ for $i \in \{1,\ldots,m\}$. If $x_1,\ldots,x_{m-1}$ all belong to A, then

$$c'+1 \leq x_m = \sum_{i=1}^{m-2} x_i + ax_{m-1} - c \leq (a+m-2)c' - c \leq (a+m-1)c'.$$

Hence $x_m \in B$, and so $\chi(x_m) = 1$.

If at least one of $x_1,\ldots,x_{m-1}$ belongs to C, then

$$x_m = \sum_{i=1}^{m-2} x_i + ax_{m-1} - c \geq c' + \min C = (a+m)c' + 1.$$

Hence x_m is outside the domain of Δ. Therefore $\Delta(x_i) = 1$ for $i \in \{1, \ldots, m\}$, and so

$$x_m = \sum_{i=1}^{m-2} x_i + ax_{m-1} - c \geq (a + m - 2) \cdot \min B - c \geq (a + m - 1)c' + 1.$$

Hence $x_m \notin B$. This proves that Δ is a valid coloring of $[1, (a+m)(a+m-c-3)]$, so that $\mathscr{R}(c) \geq (a + m)(a + m - c - 3) + 1$. ☐

Theorem 6. *Let $4 \leq m \leq a$. If a, m are of the same parity and $c < -(a - 2)(a + m - 3)$, we have*

$$\mathscr{R}(a, c; m) = (a + m)(a + m - c - 3) + 1.$$

Proof. Let $a' = a + m - 3$ throughout this proof. By Theorem 5, it is enough to show that $\mathscr{R}(a, c; m) \leq (a + m)(a' - c) + 1$. We show that every 2-coloring χ of $[0, (a+m)(a'-c)]$ exhibits a monochromatic solution to the translated Equation (2). Without loss of generality, let $\chi(0) = 0$, and by way of contradiction, assume χ is a valid coloring. The following sequences forces a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2).

- $x_i = 0$ for $1 \leq i \leq m - 1$ imply $\chi(a' - c) = 1$.
- $x_i = a' - c$ for $1 \leq i \leq m - 1$ imply $\chi\big((a + m - 1)(a' - c)\big) = 0$.
- $x_i = 0$ for $2 \leq i \leq m-1$, $x_m = (a+m-1)(a'-c)$ imply $\chi\big((a+m-2)(a'-c)\big) = 1$.
- $x_1 = (a+m-1)(a'-c)$, $x_i = 0$ for $2 \leq i \leq m-1$ imply $\chi\big((a+m)(a'-c)\big) = 1$.
- $x_i = a' - c$ for $2 \leq i \leq m - 1$, $x_m = (a + m)(a' - c)$ imply $\chi\big(2(a' - c)\big) = 0$.
- $x_1 = 2(a' - c)$, $x_i = 0$ for $2 \leq i \leq m - 1$ imply $\chi\big(3(a' - c)\big) = 1$.

We capture this information in the table below.

0	**1**
0	$a' - c$
$2(a' - c)$	$3(a' - c)$
$(a + m - 1)(a' - c)$	$(a + m - 2)(a' - c)$
	$(a + m)(a' - c)$

Table 2: Some initial colorings

We divide the proof into two cases: (i) $\chi(0) = \chi(1)$, and (ii) $\chi(0) \neq \chi(1)$.

`Case (i):` $\chi(0) = \chi(1)$. We claim that $\chi(n) = 0$ for $1 < n \leq a' - c - 1$. To do this, we show that $\chi(n) = 0$ for $1 < n \leq a - 1$ and that $\chi(m) = \chi(n)$ if $m \equiv n \pmod{a}$

and $1 < m, n \leq a' - c - 1$. Assume, by way of contradiction, that $\chi(n) = 1$ for some $n \in \{2, \ldots, a-1\}$. We claim that this forces

$$\chi\big(k(a'-c)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a+m-2]$.

We use induction on k. The base cases, $k \in \{0, 1, 2, 3\}$, are covered by the arguments in the above paragraph; see Table 2. Suppose $\chi\big(k(a'-c)\big) \equiv k \pmod 2$ for $k \in \{0, 1, 2, \ldots, K\}$ for some odd $K < a+m-2$. When k is odd, $k > 1$, the following sequences force a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2).

- $x_1 = k(a'-c)$, $x_i = n$ for $2 \leq i \leq m-1$ imply $\chi\big((k+1)(a'-c)+(a+m-3)n\big) = 0$.
- $x_1 = (k+1)(a'-c) + (a+m-3)n$, $x_i = 0$ for $2 \leq i \leq m-1$ imply $\chi\big((k+2)(a'-c) + (a+m-3)n\big) = 1$.
- $x_i = n$ for $2 \leq i \leq m-1$, $x_m = (k+2)(a'-c) + (a+m-3)n$ imply $\chi\big((k+1)(a'-c)\big) = 0$.
- $x_1 = (k+1)(a'-c)$, $x_i = 0$ for $2 \leq i \leq m-1$ imply $\chi\big((k+2)(a'-c)\big) = 1$.

Therefore, for odd k, $\chi\big(k(a'-c)\big) = 1$ implies $\chi\big((k+1)(a'-c)\big) = 0$ and $\chi\big((k+2)(a'-c)\big) = 1$. Since $\chi(a'-c) = 1$ (refer Table 2), the proof of our claim is complete.

Thus $\chi\big((a+m-2)(a'-c)\big) = 0$, contradicting the results in Table 2. This shows that $\chi(n) = 0$ for $1 < n \leq a-1$. However, in order to show that $\chi\big((a+m-2)(a'-c)\big) = 0$ we need $\chi\big((a+m-1)(a'-c) + (a+m-3)n\big) = 1$ and $(a+m-1)(a'-c) + (a+m-3)n < (a+m)(a'-c)$. Thus we must have $a'-c > (a+m-3)(a-1)$. We next show that $\chi(m) = \chi(n)$ if $m \equiv n \pmod a$ and $1 < m, n \leq a'-c-1$. We assume $\chi(n) = 0$.

- $x_1 = 1$, $x_i = 0$ for $2 \leq i \leq m-1$ imply $\chi\big(1 + (a'-c)\big) = 1$.
- $x_1 = n$, $x_i = 0$ for $2 \leq i \leq m-1$ imply $\chi\big(n + (a'-c)\big) = 1$.
- $x_1 = n + (a'-c)$, $x_i = a'-c$ for $2 \leq i \leq m-2$, $x_{m-1} = 1 + (a'-c)$ imply $\chi\big(n + a + (a+m-1)(a'-c)\big) = 0$.
- $x_i = 0$ for $2 \leq i \leq m-1$, $x_m = n + a + (a+m-1)(a'-c)$ imply $\chi\big(n + a + (a+m-2)(a'-c)\big) = 1$.
- $x_i = a'-c$ for $2 \leq i \leq m-1$, $x_m = n + a + (a+m-2)(a'-c)$ imply $\chi(n+a) = 0$.

We now have $x_1 = -c$, $x_i = 0$ for $2 \le i \le m-2$, $x_{m-1} = 1$, $x_m = 2(a'-c)$ is a monochromatic solution to Equation (2).

`Case (ii):` $\chi(0) \neq \chi(1)$. We claim that

$$\chi\big(k(a'-c)\big) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd,} \end{cases}$$

for $k \in [1, a]$. We use induction on k. The base cases, $k \in \{0,1,2,3\}$, are covered by the arguments in the above paragraph; see Table 2. Suppose $\chi\big(k(a'-c)\big) \equiv k \pmod 2$ for $k \in \{0,1,2,\ldots,K\}$ for some odd $K < a$. When k is odd, the following sequences force a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2).

- $x_1 = k(a'-c)$, $x_i = a'-c$ for $2 \le i \le m-2$, $x_{m-1} = 1$ imply $\chi\big(a + (k+m-2)(a'-c)\big) = 0$.
- $x_1 = a + (k+m-2)(a'-c)$, $x_i = 0$ for $2 \le i \le m-1$ imply $\chi\big(a + (k+m-1)(a'-c)\big) = 1$.
- $x_i = a'-c$ for $2 \le i \le m-2$, $x_{m-1} = 1$, $x_m = a + (k+m-1)(a'-c)$ imply $\chi\big((k+1)(a'-c)\big) = 0$.
- $x_1 = (k+1)(a'-c)$, $x_i = 0$ for $2 \le i \le m-1$ imply $\chi\big((k+2)(a'-c)\big) = 1$.

Therefore, for odd k, $\chi\big(k(a'-c)\big) = 1$ implies $\chi\big((k+1)(a-c)\big) = 0$ and $\chi\big((k+2)(a'-c)\big) = 1$. Since $\chi(a'-c) = 1$ (refer Table 2), the proof of our claim is complete.

If a, m are both odd, then $x_1 = (a-1)(a'-c)$, $x_2 = (m-1)(a'-c)$, $x_i = 0$ for $3 \le i \le m-1$, $x_m = (a+m-1)(a'-c)$ is a monochromatic solution to Equation (2). Therefore, any 2-coloring of $\big[0, (a+m)(a'-c)\big]$ must admit a monochromatic solution of Equation (2).

If a, m are both even, then $x_1 = a(a'-c)$, $x_2 = (m-2)(a'-c)$, $x_i = 0$ for $3 \le i \le m-1$, $x_m = (a+m-1)(a'-c)$ is a monochromatic solution to Equation (2).

Therefore, any 2-coloring of $\big[0, (a+m)(a'-c)\big]$ must admit a monochromatic solution of Equation (2), and so any 2-coloring of $\big[1, (a+m)(a'-c)+1\big]$ must admit a monochromatic solution of Equation (1) in both cases. Hence, we obtain the upper bound. □

We end this section by conjecturing that the formula in Theorem 6 extends to $c \le 0$.

Conjecture 1. Let $4 \le m \le a$ and $c \le 0$. Then

$$\mathscr{R}(a, c; m) = (a+m)(a+m-c-3) + 1,$$

provided $\mathscr{R}(a, c; m)$ exists.

4. The Case $c > a + m - 3$

We deal with the case $c > a + m - 3$ in this section. We first single out the special case where c is a multiple of $a + m - 3$. Theorem 2 and Theorem 3 provide upper bounds for $\mathscr{R}(a, c; m)$ when $a + m$ is even, and in some cases where c is even and $a + m$ is odd. We provide a lower bound for $\mathscr{R}(a, c; m)$ in this section, so that we have both lower and upper bounds in cases where $\mathscr{R}(a, c; m)$ exists. We further show this lower bound is also an upper bound when $c > (a + \kappa)(a + m - 3)$ (κ is defined in Theorem 9), thereby achieving the exact value of $\mathscr{R}(a, c; m)$ in these cases. We close the section by providing a better lower bound for $\mathscr{R}(a, c; m)$ that cover most of the integers c lying between $a + m - 3$ and $(a + \kappa)(a + m - 3)$.

Theorem 7. *Let $4 \le m \le a$ and $c = k(a + m - 3)$, $1 < k \le a + m - 2$. Then*

$$\mathscr{R}(a, c; m) = k.$$

Proof. The coloring $\Delta : [1, k - 1] \to \{0, 1\}$, defined by $\Delta(x) = 0$ for $x \in [1, k - 1]$, is a valid coloring since

$$\begin{aligned}\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m &\le (a + m - 2)(k - 1) - 1 \\ &= k(a + m - 3) + (k - 2) - (a + m - 3) \\ &< k(a + m - 3).\end{aligned}$$

Hence $\mathscr{R}(a, c; m) \ge k$.

On the other hand, since $x_1 = \cdots = x_m = k$ satisfies Equation (1) for $c = k(a + m - 3)$, every coloring $\chi : [1, k] \to \{0, 1\}$ admits a monochromatic soloution to Equation (1). Hence $\mathscr{R}(a, c; m) \le k$. $\square$

Theorem 8. *Let $4 \le m \le a$ and $c > a + m - 3$. Then*

$$\mathscr{R}(a, c; m) \ge \left\lceil \frac{1 + c(a + m)}{1 + (a + m)(a + m - 3)} \right\rceil.$$

Proof. For convenience, we set

$$\kappa = \left\lceil \frac{1 + c(a + m)}{1 + (a + m)(a + m - 3)} \right\rceil - 1,$$

and show that

$$\kappa(a + m - 2)^2 - c(a + m - 1) < \kappa(a + m - 2) - c < \kappa. \tag{3}$$

Both inequalities are equivalent to

$$\kappa < \frac{c}{a + m - 3}.$$

From the definition of κ, since $c > a+m-3$, we have

$$\kappa < \frac{1+c(a+m)}{1+(a+m)(a+m-3)} < \frac{c}{a+m-3}. \tag{4}$$

Thus both inequalities in Inequation (3) hold. Now

$$\begin{aligned} \kappa(a+m-2)-c &\geq (a+m-2)\left(\frac{1+c(a+m)}{1+(a+m)(a+m-3)}-1\right)-c \\ &= \frac{c(a+m-1)-(a+m)(a+m-2)(a+m-3)}{1+(a+m)(a+m-3)} \\ &> 0 \end{aligned}$$

if $c \geq (a+m-1)(a+m-3)$.

Let $\Delta : [1,\kappa] \to \{0,1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in \big(\max\{0, \kappa(a+m-2)^2 - c(a+m-1)\}, \kappa(a+m-2)-c\big]; \\ 1 & \text{otherwise.} \end{cases}$$

We claim that Δ provides a valid 2-coloring of $[1,\kappa]$ with respect to Equation (1). If $\Delta(x_i) = 0$ for $1 \leq i \leq m-1$, then

$$x_m \leq (a+m-2)\big(\kappa(a+m-2)-c\big)-c = \kappa(a+m-2)^2 - c(a+m-1).$$

Hence $\Delta(x_m) \neq 0$. Therefore, we must have $\Delta(x_i) = 1$ for $i \in \{1,\ldots,m\}$.

If $x_i > \kappa(a+m-2)-c$ for $i \in \{1,\ldots,m-1\}$, then

$$\begin{aligned} \kappa(a+m-2)^2 - c(a+m-1) &= (a+m-2)\big(\kappa(a+m-2)-c\big)-c \\ &< \sum_{i=1}^{m-2} x_i + a x_{m-1} - c \\ &= x_m \\ &\leq \kappa(a+m-2)-c. \end{aligned}$$

Therefore, $x_i \in \big[1, \kappa(a+m-2)^2 - c(a+m-1)\big]$ for at least one $i \in \{1,\ldots,m-1\}$. Now

$$\begin{aligned} x_m &\leq \kappa(a+m-2)^2 - c(a+m-1) + \kappa(a+m-3) - c \\ &= \kappa\big(1+(a+m)(a+m-3)\big) - c(a+m) \\ &\leq 0, \end{aligned}$$

so that x_m is outside the domain of Δ. We note that $\kappa(a+m-2)^2 - c(a+m-1) < \kappa(a+m-2)-c$ for $c > a+m-3$, and further that $\kappa(a+m-2)-c > 0$ if $c > (a+m-1)(a+m-3)$. Thus Δ provides a valid 2-coloring for $c > (a+m-1)(a+m-3)$.

For $c \in [a+m-2, (a+m-2)^2)$, it may be the case that $\kappa(a+m-2)-c<1$, in which case all integers in the interval $[1,\kappa]$ are colored 1. Since

$$x_m = \sum_{i=1}^{m-2} x_i + ax_{m-1} - c \leq \kappa(a+m-2) - c,$$

Δ provides a valid 1-coloring if $\kappa(a+m-2)-c<1$. Therefore, $\mathscr{R}(a,c;m) > \kappa$ in any case. □

Theorem 9. *Let $4 \leq m \leq a$ and let a, m be of the same parity. Then*

$$\mathscr{R}(a,c;m) = \kappa + 1 \ \text{ if } \ c > (a+\kappa)(a+m-3),$$

where

$$\kappa = \left\lceil \frac{1+c(a+m)}{1+(a+m)(a+m-3)} \right\rceil - 1.$$

Proof. Let κ be defined as in Theorem 8, and let $c > (a+m-1)(a+m-3)$. Then

$$\kappa = \left\lceil \frac{1+c(a+m)}{1+(a+m)(a+m-3)} \right\rceil - 1 \geq \frac{(c-a-m+3)(a+m)}{1+(a+m)(a+m-3)} > a+m-3. \quad (5)$$

Since $c > (a+\kappa)(a+m-3)$ by assumption, by Theorem 8, it suffices to prove that

$$\mathscr{R}(c) \leq \left\lceil \frac{1+c(a+m)}{1+(a+m)(a+m-3)} \right\rceil = \kappa + 1.$$

Let $\chi : [1, \kappa+1] \to \{0,1\}$ be any 2-coloring of the integers in the interval $[1,\kappa+1]$. Consider the complimentary coloring $\overline{\chi} : [1,\kappa+1] \to \{0,1\}$ given by

$$\overline{\chi}(x) = \chi(\kappa+2-x).$$

Then monochromatic solutions to Equation (1) under χ correspond to monochromatic solutions to

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = (\kappa+2)(a+m-3) - c \quad (6)$$

under $\overline{\chi}$.

From Inequation (4), $(\kappa+2)(a+m-3)-c < (a+m-3)\left(\frac{c}{a+m-3}+2\right) - c = 2(a+m-3)$. Thus we have $c' = (\kappa+2)(a+m-3)-c < 2(a+m-3)$ for $c > (a+m-1)(a+m-3)$. If $c' < -(a+m-3)(a-2)$, then every 2-coloring of $[1, (a+m-c'-3)(a+m)+1]$ admits a monochromatic solution to Equation (6) by Theorem 6. Now using Inequation (5) we have

$$\begin{aligned}
(a+m-c'-3)(a+m)+1 &= \big(c-(\kappa+1)(a+m-3)\big)(a+m)+1 \\
&= 1+c(a+m) - \big(1+(a+m)(a+m-3)\big)(\kappa+1) \\
&\quad +\kappa+1 \\
&\leq \kappa+1.
\end{aligned}$$

Hence, every 2-coloring of $[1, \kappa+1]$ admits a monochromatic solution to Equation (1) in this case. $\square$

Theorem 10. *Let* $4 \le m \le a$ *and* $c > a+m-3$. *Let* $c = \lambda(a+m-3) - \mu$, *with* $3 \le \lambda \le a+m-2$ *and* $1 \le \mu \le a+m-2-\lambda$. *Then*

$$\mathscr{R}(a, c; m) \ge \lambda + \mu.$$

Proof. Let $\Delta : [0, \lambda+\mu-2] \to \{0, 1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in [0, \lambda-2]; \\ 1 & \text{if } x \in [\lambda-1, \lambda+\mu-2]. \end{cases}$$

We claim that Δ provides a valid 2-coloring of $[0, \lambda+\mu-2]$ with respect to Equation (2). Suppose $\Delta(x_1) = \cdots = \Delta(x_{m-1}) = 0$. Then

$$\begin{aligned} x_m &= \sum_{i=1}^{m-2} x_i + a x_{m-1} - (\lambda-1)(a+m-3) + \mu \\ &\le (\lambda-2)(a+m-2) - (\lambda-1)(a+m-3) + \mu \\ &\le -1, \end{aligned}$$

so that x_m is outside the domain of Δ. Therefore, we must have $\Delta(x_i) = 1$ for $i \in \{1, \ldots, m\}$. But then

$$\begin{aligned} x_m &= \sum_{i=1}^{m-2} x_i + a x_{m-1} - (\lambda-1)(a+m-3) + \mu \\ &\ge (\lambda-1)(a+m-2) - (\lambda-1)(a+m-3) + \mu \ge \lambda + \mu - 1, \end{aligned}$$

and again x_m is outside the domain of Δ. Hence $\mathscr{R}(a, c; m) \ge \lambda + \mu$. $\square$

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