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An Introduction to Ramsey's Theorem

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Abstract: Ramsey's theorem is an integral part of results of the type that may loosely be classified as those that satisfy the property that if a large enough system is partitioned arbitrarily into finitely many subsystems, at least one subsystem has that particular property. Although initially stated as a result in mathematical Logic, Ramsey's theorem is now considered one of the cornerstones of Combinatorics.

Keywords: colouring; Ramsey number; graph Ramsey number; arrows into; non-complete Ramsey Theory

AMS Subject Classifications: 05D10; 05C55

`Frank Plumpton Ramsey`
(1903 – 1930)

Frank Ramsey was a British mathematician who, in addition to Mathematics, made significant contributions in Philosophy and Economics at an early age before his death at the age of 26. Frank was born on 22 February 1903 in Cambridge where his father Arthur, also a mathematician, was President of Magdalene College. He was the eldest of two brothers and two sisters, and his brother Michael, the only one of the four siblings who was to remain Christian, later became Archbishop of

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Canterbury. He entered Winchester College in 1915 and later returned to Cambridge to study Mathematics at Trinity College. With support from the economist JOHN MAYNARD KEYNES he became a Fellow of King's College, Cambridge in 1924, being the second person ever to be elected without having previously studied at King's College. In 1926 he became a University Lecturer in Mathematics and later a Director of Studies in Mathematics at King's College.

In 1927 Ramsey published the influential article *Facts and Propositions*, in which he proposed what is sometimes described as a *Redundancy Theory of Truth*. His other philosophical works include *Universals* (1925), *Universals of Law and of Fact* (1928), *Knowledge* (1929), *Theories* (1929), and *General Propositions and Causality* (1929). The philosopher LUDWIG WITTGENSTEIN, whose work *Tractatus Logico-Philosophicus* he helped translate into English, mentions him in the introduction to his *Philosophical Investigations* as an influence.

Ramsey's three papers in Economics were on *Subjective Probability and Utility* (1926), *Optimal Taxation* (1927) and *Optimal One–sector Economic Growth* (1928). The economist PAUL SAMUELSON described them in 1970 as "*three great legacies – legacies that were for the most part mere by-products of his major interest in the foundations of mathematics and knowledge.*"

One of the theorems proved by Ramsey in his 1930 paper "*On a problem of formal logic*" now bears his name. While this theorem is the work Ramsey is probably best remembered for, he only proved it in passing, as a minor lemma along the way to his true goal in the paper – solving a special case of the *Decision Problem for First–order Logic*, namely the *Decidability of Bernays-Scönfinkel–Ramsey Class of First–order Logic*. A great amount of later work in Mathematics was fruitfully developed out of the ostensibly minor lemma, which turned out to be an important early result in Combinatorics, supporting the idea that within some sufficiently large systems, however disordered, there must be some order.

Easy going, simple and modest, Ramsey had many interests besides his scientific work. He was immensely widely read in English literature; he enjoyed classics though he was on the verge of plunging into being a mathematical specialist. He was very interested in politics, and well–informed; he had got a political concern and a sort of left-wing caring–for–the–underdog kind of outlook about politics.

Suffering from chronic liver problems, Ramsey contracted jaundice after an abdominal operation and died on 19 January 1930 at Guy's Hospital in London a month before turning 27.

The Decision Analysis Society annually awards the *Frank P. Ramsey Medal* to recognise substantial contributions to Decision Theory and its application to important classes of real decision problems.

1. Ramsey's Theorem

The newest of the three major results on Ramsey–type theorems – the theorem of Ramsey in Combinatorics that bears his name – was enunciated as a result in Logic. Ramsey's Theorem may be considered as a refinement of the *Pigeonhole Principle*, but one in which we are not only guaranteed a certain number of elements in a particular class but also guaranteed that these elements share a given property. The following problem, considered folklore, amply describes this situation.

PROBLEM 1.1 **(The Party Problem)**
At a party consisting of six persons, there must be three mutual acquaintances or three mutual strangers.

A simple application of the *Pigeonhole Principle* provides a proof of this problem. Consider the complete graph $\mathcal{K}_6$, with vertices $0, \ldots, 5$, each representing a party–goer, and colour the edges between acquaintances blue and those between strangers red. By a triangle we mean those three–sided figures with all vertices $0, \ldots, 5$. Thus, a triangle with all sides blue will depict the situation where the three vertices represent persons who are mutual acquaintances, and a triangle with all sides red will depict the situation where the three vertices represent persons who are mutual strangers. A proof must consist of showing that no matter how we colour each edge in one of two colurs blue, red, one of these two monochromatic triangles must arise. By the Pigeonhole Principle, at least three of the edges 01, 02, 03, 04, 05 must be of one colour, say blue. By renumbering, if necessary, suppose the edges 01, 02, and 03 are colored blue. If any one of the edges 12, 23, 13 is coloured blue, then the triangle with vertices 0 and the two endpoints of the blue edge form a blue triangle. If none of the edges 12, 23, 13 is coloured blue, then the triangle with vertices 1, 2, 3 form a red triangle. If three of the edges 01, 02, 03, 04, 05 are coloured red, the same argument with the roles of blue and red interchanged again results in a monochromatic triangle.

The mathematical statement captured by this statement of this problem is $\mathcal{R}(3,3) \leq 6$. The two 3's represent the two relationships (acquaintances, strangers) or the two colour classes (blue, red), whereas the 6 represents that fact that six people suffice to capture one or the other situation. More generally, given positive integers m, n, the statement

$$\mathcal{R}(m,n) = N$$

is the combination of the following two statements:

- If all the edges of $\mathcal{K}_N$ are coloured either blue or red in any manner, then there must exist m vertices such that all the edges formed by the graph $\mathcal{K}_m$ on these vertices are coloured blue, or there must exist n vertices such that all the edges formed by the graph $\mathcal{K}_n$ on these vertices are coloured red, and
- There is a colouring of the edges of $\mathcal{K}_{N-1}$ in blue and red such that neither of the two situations listed above arises.

The first of these situations is captured by the statement $\mathcal{R}(m,n) \leq N$, and the second by $\mathcal{R}(m,n) > N-1$. Therefore, together these imply $\mathcal{R}(m,n) = N$. Note that the roles of blue and red are interchangeable. Hence $\mathcal{R}(m,n) = \mathcal{R}(n,m)$, and it is customary to use $\mathcal{R}(m,n)$ with $m \geq n \geq 1$. It is trivial that $\mathcal{R}(m,1) = 1$. To see why $\mathcal{R}(m,2) = m$, colouring all edges of $\mathcal{K}_{m-1}$ blue simultaneously avoids a blue $\mathcal{K}_m$ and a red $\mathcal{K}_2$, and hence implies $\mathcal{R}(m,2) > m-1$. On the other hand, the only way to avoid a red $\mathcal{K}_2$ in a blue–red edge colouring of $\mathcal{K}_m$ is by colouring all edges blue, in which case there is a blue $\mathcal{K}_m$. Hence there is always either a blue $\mathcal{K}_m$ or a red $\mathcal{K}_2$ in every blue–red edge colouring of $\mathcal{K}_m$, implying that $\mathcal{R}(m,2) \leq m$, so that $\mathcal{R}(m,2) = m$. The nontrivial values of $\mathcal{R}(m,n)$ therefore start with $m \geq n \geq 3$, and $\mathcal{R}(3,3)$ is the first of these. The choice of the Party Problem as an initial example mentioned at the start of this section is therefore quite natural.

The proof of the Party Problem implies $\mathcal{R}(3,3) \leq 6$. In fact, it is true that $\mathcal{R}(3,3) = 6$. If we colour the outer five edges of $\mathcal{K}_5$ blue and the inner diagonals red, we find no triangle of the same colour. This solitary example of 2-colouring the edges of $\mathcal{K}_5$ shows that $\mathcal{R}(3,3) > 5$, and hence also that $\mathcal{R}(3,3) = 6$. The numbers $\mathcal{R}(m,n)$ are the simplest examples of Ramsey numbers. Their existence is guaranteed by the following theorem.

THEOREM 1.2 *The Ramsey numbers $\mathcal{R}(m,n)$ satisfy the recurrence*

$$\mathcal{R}(m,n) \leq \mathcal{R}(m-1,n) + \mathcal{R}(m,n-1)$$

for $m,n \geq 2$. Moreover, if both $\mathcal{R}(m-1,n)$ and $\mathcal{R}(m,n-1)$ are even, we have

$$\mathcal{R}(m,n) \leq \mathcal{R}(m-1,n) + \mathcal{R}(m,n-1) - 1.$$

The Ramsey numbers $\mathcal{R}(m,n)$ satisfy the bounds

$$(m-1)(n-1) + 1 \leq \mathcal{R}(m,n) \leq \binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}$$

for $m,n \geq 2$.

Proof. Let us write $N = \mathcal{R}(m-1,n) + \mathcal{R}(m,n-1)$ for convenience. To prove the general upper bound, we must show that in any blue–red colouring of the edges of $\mathcal{K}_N$, there must exist either a blue $\mathcal{K}_m$ or a red $\mathcal{K}_n$.

Let V and E denote the set of vertices and edges, respectively, of $\mathcal{K}_N$, and consider any blue–red colouring of the edges of $\mathcal{K}_N$. Choose any $v \in V$, and partition the set $V \setminus \{v\}$ into sets $B = \{x \in V : xv \in E \text{ and is coloured blue}\}$ and $R = \{x \in V : xv \in E \text{ and is coloured red}\}$. Then $|B| + |R| = N - 1 = \mathcal{R}(m-1,n) + \mathcal{R}(m,n-1) - 1$, so that $|B| < \mathcal{R}(m-1,n)$ and $|R| < \mathcal{R}(m,n-1)$ is not simultaneously possible. Therefore at least one of $|B| \geq \mathcal{R}(m-1,n)$ and $|R| \geq \mathcal{R}(m,n-1)$ must hold.

Consider the case $|B| \geq \mathcal{R}(m-1,n)$; the parallel case $|R| \geq \mathcal{R}(m,n-1)$ can be argued by replacing the role of blue with red. Since the subgraph of $\mathcal{K}_B$ of $\mathcal{K}_N$ has at least $\mathcal{R}(m-1,n)$ vertices, $\mathcal{K}_B$ must contain either a blue $\mathcal{K}_{m-1}$ or a red $\mathcal{K}_n$ by definition of Ramsey number $\mathcal{R}(m-1,n)$. If the first of these cases hold, then the vertex v together with those of $\mathcal{K}_{m-1}$ forms a blue $\mathcal{K}_m$ by construction of B. Thus, in any case, $\mathcal{K}_N$ must contain either a blue $\mathcal{K}_m$ or a red $\mathcal{K}_n$. This completes the assertion that $\mathcal{R}(m,n) \leq \mathcal{R}(m-1,n) + \mathcal{R}(m,n-1)$ for $m,n \geq 2$.

To prove the stronger upper bound in the special case where both $\mathcal{R}(m-1,n)$ and $\mathcal{R}(m,n-1)$ are even, consider any blue–red colouring of the edges of $\mathcal{K}_{N-1}$ and choose a vertex $v \in V$ of even degree; this choice is made possible because the sum of degrees of all vertices in a graph equals twice the number of edges in the graph and $N-1$ is odd. With B and R as defined earlier, we now have $|B| + |R| = N - 2$. If $|B| \geq \mathcal{R}(m-1,n)$, the earlier argument implies $\mathcal{K}_{N-1}$ must contain a blue $\mathcal{K}_m$. Otherwise, $|R| \geq \mathcal{R}(m,n-1)$ since $\deg v$ is even, and again the earlier argument implies $\mathcal{K}_{N-1}$ must contain a red $\mathcal{K}_n$.

The proof of the upper bound the Ramsey numbers $\mathcal{R}(m,n)$ may be accomplished by induction on $k = m+n$. We may easily verify the bound for all cases where $k \leq 5$, since $\mathcal{R}(m,1) = 1$ and $\mathcal{R}(m,2) = 2$. For the same reason we may also assume $m,n \geq 3$. Assume the bound holds for all pairs of positive integers m,n with $m+n < k$ and $m,n \geq 3$, and consider $\mathcal{R}(m,n)$ where $m+n = k$, $m,n \geq 3$.

By inductive hypothesis

$$\mathcal{R}(m-1,n) \leq \binom{m+n-3}{m-2} \text{ and } \mathcal{R}(m,n-1) \leq \binom{m+n-3}{m-1}.$$

Applying the recurrence satisfied by the Ramsey numbers $\mathcal{R}(m,n)$ yields

$$\mathcal{R}(m,n) \leq \mathcal{R}(m-1,n)+\mathcal{R}(m,n-1) \leq \binom{m+n-3}{m-2}+\binom{m+n-3}{m-1} = \binom{m+n-2}{m-1}.$$

This completes the proof of the upper bound for $\mathcal{R}(m,n)$ by induction.

To prove the lower bound, we need to 2-colour the edges of $\mathcal{K}_{(m-1)(n-1)}$ such that there is no blue $\mathcal{K}_{m-1}$ and there is no red $\mathcal{K}_{n-1}$. Place the $(m-1)(n-1)$ vertices of $\mathcal{K}_{(m-1)(n-1)}$ in a rectangular array in $m-1$ rows and $n-1$ columns. Colour any two vertices in the same row red, and in different rows blue. Then the red edges form $m-1$ copies of $\mathcal{K}_{n-1}$, and so there is no red $\mathcal{K}_n$. There are also no blue $\mathcal{K}_m$, for among any m vertices two must be in the same row and must be joined by a red edge. Therefore the given blue-red colouring has neither a blue $\mathcal{K}_m$ nor a red $\mathcal{K}_n$. This completes the proof of the lower bound for $\mathcal{R}(m,n)$ by an example. ■

Theorem 1.2 gives $\mathcal{R}(m,3) \leq \frac{1}{2}(m^2+m)$ for $m \geq 3$. This upper bound can be improved quite easily to $\mathcal{R}(m,3) \leq \frac{1}{2}(m^2+3)$ for $m \geq 3$ by induction. However, actual rate of growth for the Ramsey numbers $\mathcal{R}(m,3)$ is $m^2/\log m$ for large m.

Theorem 1.3 **([1, 18])** *There exist constants c_1 and c_2 such that*

$$c_1 \frac{m^2}{\log m} \leq \mathcal{R}(m,3) \leq c_2 \frac{m^2}{\log m}.$$

The lower bound is due to Kim [18]; the upper bound to Ajtai, Komlós and Szemerédi [1].

The diagonal Ramsey numbers $\mathcal{R}(n,n)$ have received considerable attention. The upper bound $\mathcal{R}(n,n)$ from Theorem 1.2 is $\binom{2n-2}{n-1}$; this is asymptotically $c4^n/\sqrt{n}$. For the lower bound, the following theorem, due to Erdős, is asymptotically sharp. This proof is significant also because probabilistic methods were introduced for the first time in Ramsey theory.

Theorem 1.4 **([9])**

$$\mathcal{R}(n,n) > (e\sqrt{2})^{-1} n 2^{n/2}\big(1+o(1)\big).$$

Proof. We sketch a proof of the weaker lower bound $\mathcal{R}(n,n) > 2^{(n-2)/2}$.

Let N be a positive integer, which is to be specified later and which will serve as a lower bound. Let the vertices of $\mathcal{K}_N$ be labelled $1,2,3,\ldots,N$, and randomly colour all edges of $\mathcal{K}_N$ either red or blue, independently and with equal probability 1/2. Consider any n-subset X of $[N]$. There are $\binom{n}{2}$ edges in X; the probability that all are coloured either red or blue is $2^{-\binom{n}{2}}$. Therefore the probability that all edges in X have the same colour is $2 \cdot 2^{-\binom{n}{2}}$. Since there are $\binom{N}{n}$ ways of choosing n-subsets of $[N]$, the total probability that there exists a monochromatic n-subset of $[N]$ is $\binom{N}{n} \cdot 2^{1-\binom{n}{2}}$.

For fixed n, if we choose N such that this probability $\binom{N}{n} \cdot 2^{1-\binom{n}{2}}$ is less than 1, then we must have a colouring which contains no monochromatic n-set. The weak estimates

$$\binom{N}{n} < N^n \quad \text{and} \quad 1 - \binom{n}{2} < -\frac{n(n-2)}{2} \tag{1}$$

yield

$$\binom{N}{n} \cdot 2^{1-\binom{n}{2}} < N^n \cdot 2^{-n(n-2)/2} = \left(N \cdot 2^{-(n-2)/2}\right)^n .$$

Thus, the probability $\binom{N}{n} \cdot 2^{1-\binom{n}{2}}$ is less than 1 if $N = 2^{(n-2)/2}$.

The bound in the theorem is a consequence of applying stronger bounds in eqn. (1) via Stirling's formula:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \mathrm{O}\left(\frac{1}{n}\right)\right).$$

■

Putting together the two bounds for $\mathcal{R}(n,n)$ leads to

$$\sqrt{2} \leq \liminf \sqrt[n]{\mathcal{R}(n,n)} \leq \limsup \sqrt[n]{\mathcal{R}(n,n)} \leq 4. \tag{2}$$

Problem 1.5 **(Open Problem)**

- *Does* $\lim \sqrt[n]{\mathcal{R}(n,n)}$ *exist?*
- *Determine* $\lim \sqrt[n]{\mathcal{R}(n,n)}$, *if it exists.*

$n \backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 42	47 50	53 59	60 68	67 77	74 87
4		18	25	36 41	49 61	59 84	73 115	92 149	102 191	128 238	138 291	147 349	155 417
5			43 48	58 87	80 143	101 216	133 316	149 442	183 633	203 848	233 1138	267 1461	269 1878
6				102 165	115 298	134 495	183 780	204 1171	256 1804	294 2566	347 3703	5033	401 6911
7					205 540	217 1031	252 1713	292 2826	405 4553	417 6954	511 10578	15263	22112
8						282 1870	329 3583	343 6090	10630	16944	817 27485	41525	865 63609
9							565 6588	581 12677	22325	38832	64864		
10								798 23556	45881	81123			1265

Table of the Ramsey numbers $\mathcal{R}(m,n)$: Exact Values & Bounds

With r parameters, $r > 2$, the definition of the Ramsey numbers $\mathcal{R}(n_1, \ldots, n_r)$ have a natural extension. The Ramsey number $\mathcal{R}(n_1, \ldots, n_r)$ denotes the least positive integer N for which the following property holds: if we colour each edge in $\mathcal{K}_N$ with one of r fixed colours randomly, there must exist a $\mathcal{K}_{n_1}$ in colour 1, or a $\mathcal{K}_{n_2}$ in colour 2, or a $\mathcal{K}_{n_3}$ in colour 3, and so on. The case $r = 2$ is obviously a special case. The following generalization of Theorem 1.2 settles the question of existence of the Ramsey numbers $\mathcal{R}(n_1, \ldots, n_r)$.

THEOREM 1.6 *The Ramsey numbers $\mathcal{R}(n_1, \ldots, n_r)$ satisfy the recurrence*

$$\mathcal{R}(n_1, \ldots, n_r) \leq \sum_{i=1}^{r} \mathcal{R}(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_r)$$

for $n_1, \ldots, n_r \geq 2$.
The Ramsey numbers $\mathcal{R}(n_1, \ldots, n_r)$ have the upper bound

$$\mathcal{R}(n_1, \ldots, n_r) \leq \binom{n_1 + \cdots + n_r - r}{n_1 - 1, \ldots, n_r - 1}$$

valid for $n_1, \ldots, n_r \geq 2$.

The general version of Ramsey's theorem is considerably more complicated. Given positive integers k and r, and sufficiently large N, each k–subset of $[N]$ is assigned one of r colours. Ramsey's theorem assures the existence of such N. More precisely, if k is any positive integer, $\ell_1, \ldots, \ell_r$ satisfy $\ell_i \geq k$ for each i, and we r–colour all k–subsets of $[N]$, for some sufficently large N, then all k–subsets of some ℓ_i numbers chosen from $[N]$ must necessarily be coloured i.

THEOREM 1.7 **(Ramsey's Theorem)**
For positive integers $k, \ell_1, \ldots, \ell_r$, with each $\ell_i \geq k$, there exists a least positive integer $N = \mathcal{R}_k(\ell_1, \ldots, \ell_r)$ such that, for every r–colouring of all k–subsets of $[N]$, there exists a monochromatic set of size ℓ_i for some $i \in [r]$.

When $\ell_1 = \cdots = \ell_r = \ell$, we write $\mathcal{R}_k(\ell; r)$ for $\mathcal{R}_k(\ell_1, \ldots, \ell_r)$. If $k = 2$, we usually suppress the subscript and write $\mathcal{R}(\ell_1, \ldots, \ell_r)$ for $\mathcal{R}_2(\ell_1, \ldots, \ell_r)$. Proof of existence of the generalized Ramsey numbers $\mathcal{R}_k(\ell_1, \ldots, \ell_r)$ is considerably harder to prove; see [15] for instance.

2. Graph Ramsey Theory

Graph Ramsey theory involves graphs in Ramsey theory, as the name suggests, and graph Ramsey numbers have graphs as inputs instead of positive integers. More specifically, given any finite collection of graphs $G_1, \ldots, G_r$, $r \geq 2$, there exists N such that every edge colouring of $\mathcal{K}_N$ in r colours contains a copy of G_1 in colour 1, or a copy of G_2 in colour 2, or a copy of G_3 in colour 3, and so on. The existence of such an N follows for the Ramsey number $\mathcal{R}(n_1, \ldots, n_r)$, where n_i denotes the number of vertices in the graph G_i, $1 \leq i \leq r$. Recall that the Ramsey number corresponding to positive integers $n_1, \ldots, n_r$ involve positive integers N for which every r-colouring of edges in $\mathcal{K}_N$ must contain a $\mathcal{K}_{n_i}$ in colour i for at least one i. The graph Ramsey number $\mathcal{R}(G_1, \ldots, G_r)$ is the least positive integer N for which the above mentioned property holds. Since each G_i is contained in $\mathcal{K}_{n_i}$, the existence of graph Ramsey numbers follow from the corresponding Ramsey numbers. In fact,

$$\mathcal{R}(G_1, \dots, G_r) \leq \mathcal{R}(n_1, \dots, n_r),$$

where n_i denotes the order of G_i, $1 \leq i \leq r$.

Graph Ramsey theory has attracted a lot of interest, specially since the late 60's. As in the case with Ramsey numbers, most of research has centered around the case $r = 2$ because of expected simplicity in the argument in this case as opposed to the cases $r > 2$. Finding exact values of graph Ramsey numbers is an extremely challenging problems, even in the case $r = 2$. For instance, the statement

$$\mathcal{R}(G_1, G_2) = N$$

is the combination of the following two statements:

- If all the edges of $\mathcal{K}_N$ are coloured either blue or red in any manner, the graph formed by considering only the blue edges must contain G_1 as a subgraph, or the graph formed by considering only the red edges must contain G_2 as a subgraph, and
- There is a colouring of the edges of $\mathcal{K}_{N-1}$ in blue and red such that neither of the two situations listed above arises.

The first of these situations is captured by the statement $\mathcal{R}(G_1, G_2) \leq N$, and the second by $\mathcal{R}(G_1, G_2) > N - 1$. Therefore, together these imply $\mathcal{R}(G_1, G_2) = N$. Note that the roles of blue and red are interchangeable.

Some of the earliest results in graph Ramsey theory include determining $\mathcal{R}(P_m, P_n)$, $\mathcal{R}(C_m, C_n)$, $\mathcal{R}(T_m, \mathcal{K}_n)$, and $\mathcal{R}(\mathcal{K}_{1,n_1}, \dots, \mathcal{K}_{1,n_r})$. Here P_n, C_n, T_n denote path, cycle, tree, respectively, each of order n, and $\mathcal{K}_{1,n}$ denotes a complete bipartite graph with partite sets of orders 1 and n, and is called a star graph.

Theorem 2.1 (**[13]**)
For integers m, n, with $2 \leq n \leq m$,

$$\mathcal{R}(P_m, P_n) = m + \lfloor \tfrac{n}{2} \rfloor - 1.$$

Theorem 2.2 (**[12, 26, 27]**)
For integers m, n, with $3 \leq n \leq m$,

$$\mathcal{R}(C_m, C_n) = \begin{cases} 2m - 1 & \text{if } n \text{ is odd, } (m,n) \neq (3,3); \\ m + \frac{n}{2} - 1 & \text{if } m, n \text{ are even, } (m,n) \neq (4,4); \\ \max\{m + \frac{n}{2} - 1, 2n - 1\} & \text{if } m \text{ is odd and } n \text{ is even}; \\ 6 & \text{if } m = n \in \{3,4\}. \end{cases}$$

Theorem 2.3 (**[6]**)
If T_m is any tree of order m and n is a positive integer, then

$$\mathcal{R}(T_m, \mathcal{K}_n) = (m - 1)(n - 1) + 1.$$

Theorem 2.4 **([4])**
Let $n_1, \ldots, n_k$ be positive integers, e of which are even. Then

$$\mathcal{R}(\mathcal{K}_{1,n_1}, \ldots, \mathcal{K}_{1,n_k}) = \begin{cases} N+1 & \text{if } e \text{ is even and positive,} \\ N+2 & \text{otherwise,} \end{cases}$$

where $N = \sum_{i=1}^{k}(n_i - 1)$.

We close this section with a sketch of the proof of Theorem 2.3.

Proof of Theorem 2.3.
To establish the lower bound $\mathcal{R}(T_m, \mathcal{K}_n) > (m-1)(n-1)$, we must exhibit a colouring of each of the edges of $\mathcal{K}_{(m-1)(n-1)}$ in red or blue for which there is no red T_m and no blue $\mathcal{K}_n$. Place the $(m-1)(n-1)$ vertices in a $(m-1) \times (n-1)$ rectangular grid, and join any two vertices in the same row by a blue edge and any two vertices in different rows by a red edge. The subgraph with blue edges form $m-1$ copies of $\mathcal{K}_{n-1}$, thereby avoiding a blue $\mathcal{K}_n$. On the other hand, any m vertices in the subgraph with red edges must contain at least two from the same row, by Pigeonhole Principle. But these two vertices must be joined by a blue edge, which is a contradiction to our assumption that we are in the red subgraph. Therefore we have exhibited a colouring of each of the edges of $\mathcal{K}_{(m-1)(n-1)}$ in red or blue for which there is no red T_m and no blue $\mathcal{K}_n$, as desired.

To establish the upper bound $\mathcal{R}(T_m, \mathcal{K}_n) \leq (m-1)(n-1)+1$, we use the following result on trees:

> If T is any tree with $k-1$ vertices and G is any graph with minimum vertex degree $\delta(G) \geq k$, then T is a subgraph of G.

Consider any colouring of the edges of $\mathcal{K}_{(m-1)(n-1)+1}$ in red or blue, and let v be any vertex in this graph. The proof we present runs on inducting on n. The base case $n = 1$ is trivial. If v has more than $(m-1)(n-2)$ neighbours along the blue edges, then there must exist a red T_m or a blue $\mathcal{K}_{n-1}$ among these, by induction hypothesis. Together with the vertex v, the graph G then must contain either a red T_m or a blue $\mathcal{K}_n$.

Otherwise, every vertex must have at most $(m-1)(n-2)$ incident blue edges, and hence at least $m-1$ incident red edges. The quoted result on trees now shows the existence of a red T_m. This completes the sketch of the proof. ■

3. Noncomplete Ramsey Theory

Noncomplete Ramsey theory generalize both classical Ramsey theory and graph Ramsey theory. For any collection of graphs $G_1, \ldots, G_r$, we say that a graph G "*arrows into*" $(G_1, \ldots, G_r)$, and write

$$G \to (G_1, \ldots, G_r), \tag{3}$$

provided any r–colouring of the edges of G yields a monochromatic spanning subgraph each of whose edges is coloured i and that contains a G_i, for some $i \in [r]$.

Otherwise stated,

$$G = F_1 \oplus \cdots \oplus F_r \Longrightarrow F_i \supseteq G_i \text{ for at least one } i \in [r].$$

The graphs $F_1, \ldots, F_r$ are spanning subgraphs of G, and are called "factors" of G. By the containment $F_i \supseteq G_i$ one means simply that G_i is a subgraph of F_i, not necessarily a spanning subgraph. Colouring the edges of G by one of r available colours induces a factorization of G, each given by the spanning subgraph with edges of one colour. Conversely, each factorizaton of G leads to a colouring of the edges of G, with one colour assigned to all edges of each factor. Thus there is a natural correspondence between factorization of G and edge–colouring of G and the two terms may be used interchangeably.

The *arrows notation* may also be used to state Ramsey's theorem 1.7 concisely. Given positive integers $\ell_1, \ldots, \ell_r, k$, with each $\ell_i \geq k$, the notation

$$N \rightarrow (\ell_1, \ldots, \ell_r)^k$$

stands for the statement of Theorem 1.7, and the least such N for which this statement holds is $\mathcal{R}_k(\ell_1, \ldots, \ell_r)$.

The main problem of *non–complete Ramsey Theory* is to characterize graphs G that *arrow into* a given collection of graphs $G_1, \ldots, G_r$.

For any collection of graphs $G_1, \ldots, G_r$, the smallest positive integer n for which $\mathcal{K}_n \rightarrow (G_1, \ldots, G_r)$ is the *graph Ramsey number* of $G_1, \ldots, G_r$, and is denoted by $\mathcal{R}(G_1, \ldots, G_r)$. Being able to characterize G resolves many problems invoving the given graphs $G_1, \ldots, G_r$. For instance, the graph Ramsey number $\mathcal{R}(G_1, \ldots, G_r)$, which is the least positive integer n such that

$$\mathcal{K}_n \rightarrow (G_1, \ldots, G_r),$$

may be easily determined from the characterization of G in eqn. (3). In particular, the case when each G_i is also a complete graph $\mathcal{K}_{\ell_i}$, the corresponding graph Ramsey number $\mathcal{R}(\mathcal{K}_{\ell_1}, \ldots, \mathcal{K}_{\ell_r})$ coincides with the Ramsey number $\mathcal{R}(\ell_1, \ldots, \ell_r)$.

One of the first instances of a solution to the main problem of characterization of G in eqn. (3) is when $G_1 = G_2 = \mathcal{K}_{1,n}$, due to Murty.

THEOREM 3.1 *Let G be a connected graph and n a positive integer. Then*

$$G \rightarrow (\mathcal{K}_{1,n}, \mathcal{K}_{1,n})$$

if and only if

(i) $\Delta(G) \geq 2n - 1$, *or*
(ii) *n is even and G is a $(2n-2)$-regular graph of odd order.*

The result of Theorem 3.1 has been generalized by Gupta, Thulasi Rangan & Tripathi [16] to $G_1 = \mathcal{K}_{1,n_1}, \ldots, G_k = \mathcal{K}_{1,n_k}$, where $n_1, \ldots, n_k$ are any k positive integers, $k \geq 2$. The characterization of G satisfying

$$G \rightarrow (\mathcal{K}_{1,n_1}, \ldots, \mathcal{K}_{1,n_k}) \tag{4}$$

is described by one of four cases, and these cases involve conditions on the graph or their regularization. A k–factor of a graph is a factor that is k–regular, and a

$\Delta(G)$–regularization of G is a $\Delta(G)$-regular graph $G^\star$ of which G is an induced subgraph.

Theorem 3.2 **([16])**
Let G be a connected graph, let $n_1, \ldots, n_k$ be positive integers of which e are even, and let $N = \sum_{i=1}^{k}(n_i - 1)$. Let $G^\star$ be the Δ-regularization of G. Then

$$G \to (\mathcal{K}_{1,n_1}, \ldots, \mathcal{K}_{1,n_k})$$

if and only if

(i) $\Delta(G) \geq N + 1$, *or*
(ii) *G is N–regular, of odd order and e is even and non-zero, or*
(iii) *G is N–regular, of even order, at least one n_i is even, and G does not have an $n_i - 1$ factor for at least one even n_i, or*
(iv) *G is not N–regular, $\Delta(G) = N$, and $G^\star \to (\mathcal{K}_{1,n_1}, \ldots, \mathcal{K}_{1,n_k})$.*

The proof of Theorem 3.2 involves several basic results that deal with characterizations of graphs that have a k-factor, such as the ones due to Tutte [30, 31] and Petersen [22], and with edge colourings, such as the one due to Vizing [32], and independently, to Gupta [17]. Even a sketch of a proof of this result is beyond the scope of this article, but we briefly indicate how Theorem 2.4 and Theorem 3.1 may be deduced from Theorem 3.2.

Theorem 3.2 implies Theorem 2.4.
Observe that $G = \mathcal{K}_{N+2}$ satisfies eqn. (4) by condition (i). To complete the proof, we need to show that $\mathcal{K}_{N+1}$ satisfies eqn. (4) if and only if e even and non-zero.

If e is even and non-zero, condition (ii) applies to $\mathcal{K}_{N+1}$. Conversely, suppose $\mathcal{K}_{N+1}$ satisfies eqn. (4). If N is even, by condition (ii), e is even and non-zero. If N is odd, by condition (iii), $\mathcal{K}_{N+1}$ does not have an $(n_i - 1)$–factor for at least one even n_i, which contradicts the well known fact that $\mathcal{K}_{2n}$ is 1–factorable for each $n \geq 1$. ■

Theorem 3.2 implies Theorem 3.1.
When $k = 2$ and $n_1 = n_2 = n$, $N + 1 = 2(n - 1) + 1 = 2n - 1$, so that part (i) in Theorem 3.1 is a direct translation of part (i) in Theorem 3.2. Part (ii) in Theorem 3.2 reduces to G being a $(2n - 2)$–regular and of odd order, with n even.

Part (iii) in Theorem 3.2 reduces to G being a $(2n-2)$–regular and of even order, with n even, such that G does not have a $(n - 1)$-factor, and part (iv) in Theorem 3.2 reduces to G being not $(2n-2)$–regular, $\Delta(G) = 2n-2$, and $G^\star \to (\mathcal{K}_{1,n}, \mathcal{K}_{1,n})$. It can be shown that neither of these cases can occur. ■

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