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Exact and approximate results on the least size of a graph with a given degree set

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ABSTRACT

The *degree set* of a finite simple graph G is the set of distinct degrees of vertices of G. A theorem of Kapoor, Polimeni & Wall asserts that the least order of a graph with a given degree set $\mathscr{D}$ is $1+\max\mathscr{D}$. Tripathi & Vijay considered the analogous problem concerning the least size of graphs with degree set $\mathscr{D}$. We expand on their results, and determine the least size of graphs with degree set $\mathscr{D}$ when (i) $\min\mathscr{D}$ divides d for each $d \in \mathscr{D}$; (ii) $\min\mathscr{D} = 2$; (iii) $\mathscr{D} = \{m, m+1, \ldots, n\}$. In addition, given any $\mathscr{D}$, we produce a graph G whose size is within $\min\mathscr{D}$ of the optimal size, giving a $(1+\frac{2}{d_1+1})$-approximation, where $d_1 = \max\mathscr{D}$.

1. Introduction

A nonincreasing sequence $a_1, \ldots, a_p$ of nonnegative integers is said to be *graphic* if there exists a simple graph G with vertices $v_1, \ldots, v_p$ such that v_k has degree a_k for each k. Any graphic sequence clearly satisfies the conditions $a_k \leq p-1$ for each k and $\sum_{k=1}^{p} a_k$ is even. However, these conditions together do not ensure that a sequence will be graphic. Necessary and sufficient conditions for a sequence of nonnegative integers to be graphic are well known; refer [1–3]. Given a graphic sequence $\mathbf{s}$ of length p, there are polynomial-time algorithms in p to construct a graph with the degree sequence $\mathbf{s}$; refer [2,3,11].

The *degree set* of a simple graph G is the set $\mathscr{D}(G)$ consisting of the distinct degrees of vertices in G. For more discussion on degree sets of graphs refer [8,9,14]. Conversely, given any set $\mathscr{D}$ of positive integers, a natural question is to investigate the set of all graphs with degree set $\mathscr{D}$, and in particular the least order and size of such graphs. We denote by $\ell_p(\mathscr{D})$ and $\ell_q(\mathscr{D})$ respectively the least order and the least size of a graph with degree set $\mathscr{D}$. The following result answers the question for the least order of a graph with the degree set $\mathscr{D}$:

Theorem 1 (*Kapoor, Polimeni & Wall [5]*). *For each nonempty finite set $\mathscr{D}$ of positive integers, there exists a simple graph G for which $\mathscr{D}(G) = \mathscr{D}$. Moreover, there is always such a graph of order $\Delta + 1$, where $\Delta = \max\mathscr{D}$, and there is no such graph of smaller order.*

Some other works extend this result to special classes of graphs, including k-connected, k-edge-connected, and unicyclic graphs [7]; unicyclic bipartite graphs [6].

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Tripathi and Vijay [13] study the analogous question for graph size, and determine $\ell_q(\mathscr{D})$ in the following special cases:

(a) $|\mathscr{D}| \leq 3$ (Theorems 2, 7)
(b) $\mathscr{D} = \{1, \ldots, n\}$ for a positive integer n (Theorem 3)
(c) $\min \mathscr{D} \geq |\mathscr{D}|$ (Theorem 4).

In this paper, we determine $\ell_q(\mathscr{D})$ when

(a) $\min \mathscr{D} \mid d \quad$ for each $d \in \mathscr{D}$
(b) $\min \mathscr{D} = 2$
(c) $\mathscr{D} = \{m, m+1, \ldots, n\}$.

Given any set $\mathscr{D}$, we also give a graph whose size is within $(\min \mathscr{D} - 1)$ of $\ell_q(\mathscr{D})$, which is a $\left(1 + \frac{2}{d_1+1}\right)$-approximation, where $d_1 = \max \mathscr{D}$.

Throughout this paper, we let $\mathscr{D} = \{d_1, \ldots, d_n\}$ be a set of positive integers arranged in decreasing order. We shall employ the notation $(d)_m$ to denote m occurrences of the integer d. We denote a typical sequence as

$$\mathbf{s} = a_1, \ldots, a_p = (d_1)_{m_1}, \ldots, (d_n)_{m_n}, \tag{1}$$

where $a_j = d_k$ if $\sum_{i=1}^{k-1} m_i < j \leq \sum_{i=1}^{k} m_i$, and each $m_k \geq 1$ with $\sum_{i=1}^{n} m_i = p$.

We shall write

$$b_t = \sum_{i=1}^{t} m_i \text{ for } 1 \leq t \leq n.$$

We call b_t the tth *breakpoint* of the sequence $\mathbf{s}$, and the set $\{b_t : 1 \leq t \leq n\}$ as the set of breakpoints of $\mathbf{s}$.

The characterization of graphic sequence due to Erdős & Gallai [1] requires verification of as many inequalities as is the order of the graph.

Theorem 2 (*Erdős & Gallai [1]*). *A sequence $\mathbf{s} = a_1, \ldots, a_p$ is graphic if and only if $\sum_{i=1}^{p} a_i$ is even and if the inequalities*

$$\sum_{i=1}^{k} a_i \leq k(k-1) + \sum_{i=k+1}^{p} \min\{a_i, k\}$$

hold for $1 \leq k \leq p$.

We also use a refined form of Theorem 2, due to Tripathi & Vijay [12] that requires verification of only as many inequalities as the number of distinct terms in the sequence.

Theorem 3 (*Tripathi & Vijay [12]*). *A sequence $\mathbf{s} = a_1, \ldots, a_p$ with the set of breakpoints $\{b_1, \ldots, b_n\}$ is graphic if and only if $\sum_{i=1}^{p} a_i$ is even and if the inequalities*

$$\sum_{i=1}^{k} a_i \leq k(k-1) + \sum_{i=k+1}^{p} \min\{k, a_i\}$$

hold for $k \in \{b_1, \ldots, b_n\}$. Moreover, the inequality need only be checked for $1 \leq k \leq t$, where t is the largest positive integer for which $a_t \geq t - 1$.

Let sequence $\mathbf{s}$ be as given by Eq. (1). We set

$$\Delta_{\mathbf{s}}(k) = k(k-1) + \sum_{i=k+1}^{p} \min\{k, a_i\} - \sum_{i=1}^{k} a_i, \quad 1 \leq k \leq p. \tag{2}$$

Note that $\mathbf{s}$ is graphic if and only if $\sum_{i=1}^{p} a_i$ is even and $\Delta_{\mathbf{s}}(k) \geq 0$ for $k \in \{b_1, \ldots, b_n\}$ by Theorem 3.

We denote the sum of the terms of the sequence $\mathbf{s}$ by $\sigma(\mathbf{s})$, and the sum of the elements of the set S by $\sigma(S)$.

2. Basic results

In this section, we give two results which form the basis of the main work in this paper. For a set $\mathscr{D}$ of positive integers, Proposition 1 shows the existence of a graphic sequence with exactly one occurrence of each element in $\mathscr{D}$, except that the smallest odd element in $\mathscr{D}$ may occur twice, depending on parity considerations. Additionally, the smallest element in $\mathscr{D}$ will occur multiple times. We also determine the least number of possible occurrences of the smallest element in $\mathscr{D}$ in any such graphic sequence. That is, Proposition 1 determines the minimum graph size with degree set $\mathscr{D}$ subject to an additional constraint: the degree sequence of the graph must be of the form just described.

To utilize this, Lemma 1 (called Splitting lemma) gives a method to reduce the number of large degree vertices in a graph without changing the graph size. This is done by replacing large degree vertices with multiple vertices of smaller degree. Suppose we are given $\mathscr{D}$ and a minimum-size graph G with degree set $\mathscr{D}$. Ideally, given any $d \in \mathscr{D} \setminus \{\min \mathscr{D}\}$, if we could use Splitting lemma to remove all but one vertices of G with degree d and replace it by several vertices of degree $\min \mathscr{D}$, we could then use Proposition 1 to determine $\ell_q(\mathscr{D})$.

Proposition 1. *Let $\mathscr{D} = \{d_1, \ldots, d_n\}$ be a set of positive integers arranged in decreasing order.*

(a) *Let $\sigma(\mathscr{D})$ be even or d_n be odd. Let $\mathbf{s} = d_1, \ldots, d_n$, and let*

$$k^\star = \operatorname*{argmax}_{1 \le k \le n-1} \left\{ \left\lceil \frac{-\Delta_{\mathbf{s}}(k)}{\min\{k, d_n\}} \right\rceil \right\}, \quad c = \max_{1 \le k \le n-1} \left\{ \left\lceil \frac{-\Delta_{\mathbf{s}}(k)}{\min\{k, d_n\}} \right\rceil \right\} = \left\lceil \frac{-\Delta_{\mathbf{s}}(k^\star)}{\min\{k^\star, d_n\}} \right\rceil .$$

Then, there exists a non-negative integer C such that the sequence $\bar{\mathbf{s}} = d_1, \ldots, d_{n-1}, (d_n)_{C+1}$ is graphic, and the least such C is given by

$$C^\star = \begin{cases} c & \text{if } d_n \text{ and } \sigma(\mathscr{D}) \text{ are even;} \\ c & \text{if } d_n \text{ is odd and } \sigma(\mathscr{D}) + cd_n \text{ is even;} \\ c+1 & \text{if } d_n \text{ and } \sigma(\mathscr{D}) + cd_n \text{ are odd.} \end{cases}$$

Moreover $\Delta_{\bar{\mathbf{s}}}(k^\star) < 2d_n$ holds for $C = C^\star$.

(b) *Let $\sigma(\mathscr{D})$ be odd and d_n be even. Let $r = \max\{i : d_i \text{ is odd}\}$, and let $\mathbf{s} = d_1, \ldots, d_{r-1}, (d_r)_2, d_{r+1}, \ldots, d_n$. Then, there exists a non-negative integer C such that the sequence obtained by appending C copies of d_n to $\mathbf{s}$, $\bar{\mathbf{s}} = d_1, \ldots, (d_r)_2, d_{r+1}, \ldots, (d_n)_{C+1}$ is graphic, and the least such C is given by*

$$k^\star = \operatorname*{argmax}_{1 \le k \le n} \left\{ \left\lceil \frac{-\Delta_{\mathbf{s}}(k)}{\min\{k, d_n\}} \right\rceil \right\}, \quad C^\star = \max_{1 \le k \le n} \left\{ \left\lceil \frac{-\Delta_{\mathbf{s}}(k)}{\min\{k, d_n\}} \right\rceil \right\} = \left\lceil \frac{-\Delta_{\mathbf{s}}(k^\star)}{\min\{k^\star, d_n\}} \right\rceil .$$

Moreover $\Delta_{\bar{\mathbf{s}}}(k^\star) < d_n$ holds for $C = C^\star$.

Proof.

(a) Let $\sigma(\mathscr{D})$ be even or d_n be odd. Let $k^\star \in \{1, \ldots, n-1\}$ be such that $c = \left\lceil \frac{-\Delta_{\mathbf{s}}(k^\star)}{\min\{k^\star, d_n\}} \right\rceil$.

Suppose for an arbitrary nonnegative integer C, $\bar{\mathbf{s}} = d_1, \ldots, d_{n-1}, (d_n)_{C+1}$, so that $b_t = t$ for $1 \le t \le n-1$ and $b_n = n + C$, where b_t is the tth breakpoint of the sequence $\bar{\mathbf{s}}$. If $C < c$, then

$$\begin{aligned} \Delta_{\bar{\mathbf{s}}}(k^\star) &= k^\star(k^\star - 1) + \left(\sum_{i=k^\star+1}^{n} \min\{k^\star, d_i\} + C \min\{k^\star, d_n\} \right) - \sum_{i=1}^{k^\star} d_i \\ &= \left(k^\star(k^\star - 1) + \sum_{i=k^\star+1}^{n} \min\{k^\star, d_i\} - \sum_{i=1}^{k^\star} d_i \right) + C \min\{k^\star, d_n\} \\ &= \Delta_{\mathbf{s}}(k^\star) + C \min\{k^\star, d_n\} \\ &\le \Delta_{\mathbf{s}}(k^\star) + \left(\left\lceil \frac{-\Delta_{\mathbf{s}}(k^\star)}{\min\{k^\star, d_n\}} \right\rceil - 1 \right) \min\{k^\star, d_n\} \\ &< \Delta_{\mathbf{s}}(k^\star) + \left(\frac{-\Delta_{\mathbf{s}}(k^\star)}{\min\{k^\star, d_n\}} \right) \min\{k^\star, d_n\} \\ &= 0. \end{aligned}$$

Hence $\bar{\mathbf{s}}$ is not graphic when $C < c$ by Eq. (2).

If $C \ge c$ and $k < n$, then

$$\begin{aligned} \Delta_{\bar{\mathbf{s}}}(k) &= k(k-1) + \left(\sum_{i=k+1}^{n} \min\{k, d_i\} + C \min\{k, d_n\} \right) - \sum_{i=1}^{k} d_i \\ &= \left(k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} - \sum_{i=1}^{k} d_i \right) + C \min\{k, d_n\} \\ &= \Delta_{\mathbf{s}}(k) + C \min\{k, d_n\} \\ &\ge \Delta_{\mathbf{s}}(k) + c \min\{k, d_n\} \\ &\ge \Delta_{\mathbf{s}}(k) + \left\lceil \frac{-\Delta_{\mathbf{s}}(k)}{\min\{k, d_n\}} \right\rceil \min\{k, d_n\} \end{aligned}$$

$$\geq \Delta_{\mathbf{s}}(k) + \left(\frac{-\Delta_{\mathbf{s}}(k)}{\min\{k, d_n\}}\right)\min\{k, d_n\}$$
$$= 0.$$

From the definition of c and Eq. (2), we have

$$c \geq -\Delta_{\mathbf{s}}(1) = d_1 - (n-1).$$

If $C \geq c$, then $n + C \geq d_1 + 1$. Thus

$$\Delta_{\bar{\mathbf{s}}}(n+C) = (n+C)(n+C-1) - \left(\sum_{i=1}^{n-1} d_i + (C+1)d_n\right) \geq (n+C)d_1 - \left(\sum_{i=1}^{n-1} d_i + (C+1)d_n\right) \geq 0.$$

Therefore, $\bar{\mathbf{s}}$ is graphic provided $\sigma(\bar{\mathbf{s}}) = \left(\sum_{i=1}^{n-1} d_i\right) + (C+1)d_n$ is even whenever $C \geq c$. If $C = c$ and either (i) $\sigma(\mathbf{s})$ and d_n are both even, or (ii) d_n is odd and $\sigma(\mathbf{s}) + cd_n$ is even, then we observe that $\sigma(\bar{\mathbf{s}})$ is even. Therefore, in these cases, $C^\star = c$. In case d_n and $\sigma(\mathbf{s}) + cd_n$ are both odd, then $\sigma(\bar{\mathbf{s}})$ is odd or even according as $C = c$ or $C = c + 1$. Hence $C^\star = c + 1$ in this remaining case.
Finally, for $C = C^\star$ we have

$$\begin{aligned}
\Delta_{\bar{\mathbf{s}}}(k^\star) &= k^\star(k^\star - 1) + \left(\sum_{i=k^\star+1}^{n} \min\{k^\star, d_i\} + C^\star \min\{k^\star, d_n\}\right) - \sum_{i=1}^{k^\star} d_i \\
&\leq \left(k^\star(k^\star - 1) + \sum_{i=k^\star+1}^{n} \min\{k^\star, d_i\} - \sum_{i=1}^{k^\star} d_i\right) + (c+1)\min\{k^\star, d_n\} \\
&< \Delta_{\mathbf{s}}(k^\star) + \left(-\frac{\Delta_{\mathbf{s}}(k^\star)}{\min\{k^\star, d_n\}} + 2\right)\min\{k^\star, d_n\} \\
&= 2\min\{k^\star, d_n\} \\
&\leq 2d_n.
\end{aligned}$$

(b) Notice that $\sigma(\mathscr{D})$ is odd implies that $\mathscr{D}$ contains at least one odd integer. Hence r is well defined. Notice also that $\sigma(\bar{\mathbf{s}})$ is even for all $C > 0$ in this case. Therefore, by Theorem 2 it is enough to show that $\bar{\mathbf{s}}$ satisfies $\Delta_{\bar{\mathbf{s}}}(k) \geq 0$, $k \in \{1, \ldots, n-1, n+C\}$ for $C = C^\star$. The rest of the argument follows along the lines of part (a), and is omitted. ■

Given a graph G, and a vertex v in G, the Splitting lemma allows the construction of a graph G' in which the vertex v is replaced by several vertices the sum of degrees of which equals the degree of v. This is illustrated in Fig. 1.

Lemma 1 (*Splitting lemma*)**.** *Let G be a graph, and let $v \in V(G)$ with $\deg v = d$. Let $(n_1, \ldots, n_r)$ be a partition of d into positive summands. Then there exists a graph G' with $V(G') = \big(V(G) \setminus \{v\}\big) \cup \{v_1, \ldots, v_r\}$ such that $\deg v_i = n_i$, $1 \leq i \leq r$ and $|E(G')| = |E(G)|$.*

Proof. Partition the d neighbours of v into sets $S_1, \ldots, S_r$ with $|S_i| = n_i$, $1 \leq i \leq r$. Form a graph G' from G by replacing the vertex v by vertices $v_1, \ldots, v_r$ such that each v_i is adjacent to the vertices of S_i. Note that $|E(G')| = |E(G)|$. ■

3. The case where min $\mathscr{D}$ divides each element of $\mathscr{D}$

In this section, we obtain $\ell_q(\mathscr{D})$ when $\min \mathscr{D}$ divides each element of $\mathscr{D}$, and in particular, when $\min \mathscr{D} = 1$.

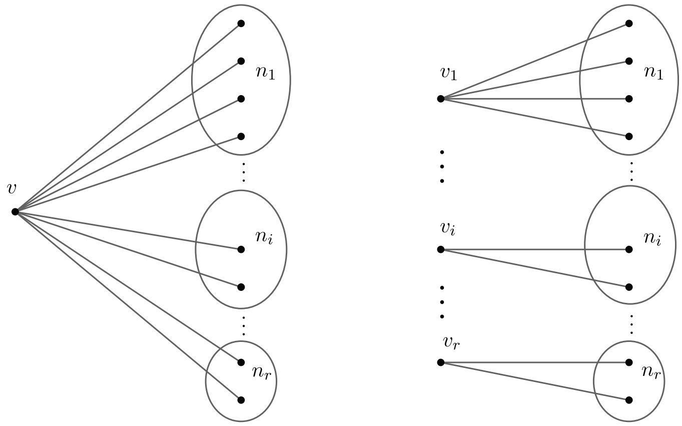
Theorem 4. *Let $\mathscr{D} = \{d_1, \ldots, d_n\}$ be a set of positive integers arranged in decreasing order such that d_n divides d for each $d \in \mathscr{D}$. Then*

$$\ell_q(\mathscr{D}) = \frac{1}{2}\Big(\sigma(\mathscr{D}) + C^\star\big(\min \mathscr{D}\big)\Big),$$

where $\sigma(\mathscr{D})$ is the sum of the elements in $\mathscr{D}$ and $C^\star$ is as defined in Proposition 1(a).

Proof. We show using Splitting lemma that there is an optimal degree sequence of the form given in Proposition 1(a). The proposition then implies the result.

Let $\mathbf{s} = (d_1)_{m_1}, (d_2)_{m_2}, \ldots, (d_n)_{m_n}$ be a minimum-size graphic sequence with degree set $\mathscr{D}$, with each $m_i \geq 1$. All but one copies of d_i, $1 \leq i < n$ can be replaced by an appropriate number of d_n's (since $d_n \mid d_i$ for each i) by Splitting lemma. Hence we arrive at a graphic sequence $\bar{\mathbf{s}} = d_1, \ldots, d_{n-1}, (d_n)_{M_n}$ for some positive integer M_n such that $\sigma(\mathbf{s}) = \sigma(\bar{\mathbf{s}})$. Therefore, there exists at least one graphic sequence of the type $\bar{\mathbf{s}} = d_1, d_2, \ldots, d_{n-1}, (d_n)_{M_n}$ for which



(a) Neighborhood of vertex v in graph G; $\deg_G(v) = n_1 + \ldots + n_r$.

(b) Vertex v is replaced by vertices $v_1, \cdots, v_r$ in graph G'; $\deg_{G'}(v_i) = n_i$.

Fig. 1. A visual representation of Splitting lemma. Vertex v is 'split' into vertices $v_1, \ldots, v_r$ without affecting the graph size.

$\sigma(\bar{\mathbf{s}}) = 2\,\ell_q(\mathscr{D})$. The minimum value of M_n such that $\bar{\mathbf{s}}$ is graphic is equal to $C^\star + 1$, as determined in Proposition 1(a). Therefore, $\ell_q(\mathscr{D}) = \frac{1}{2}\left((C^\star + 1)d_n + \sum_{i=1}^{n-1} d_i\right) = \frac{1}{2}\,(\sigma(\mathscr{D}) + C^\star(\min \mathscr{D}))$. ■

Corollary 1. *Let $\mathscr{D} = \{d_1, \ldots, d_n\}$ be a set of positive integers arranged in decreasing order such that $d_n = 1$. Then*

$$\ell_q(\mathscr{D}) = \frac{1}{2}\Big(\sigma(\mathscr{D}) + C^\star\Big),$$

where $\sigma(\mathscr{D})$ is the sum of the elements in $\mathscr{D}$, and $C^\star = \max_{1\le k\le n}\{-\Delta_{\mathbf{s}}(k)\}$, where $\mathbf{s}$ is as defined in Proposition 1(a).

Proof. This follows directly from Theorem 4. ■

4. The case $\min \mathscr{D} = 2$

In this section, we determine $\ell_q(\mathscr{D})$ when $\min \mathscr{D} = 2$. A similar argument determines $\ell_q(\mathscr{D})$ when $\min \mathscr{D} = 3$; we omit the details.

Theorem 5. *Let $\mathscr{D}$ be a set of positive integers such that $\min \mathscr{D} = 2$. Then*

$$\ell_q(\mathscr{D}) = \frac{1}{2}\Big(\sigma(\mathscr{D}) + 2C^\star\Big) + \begin{cases} 0 & \text{if } \sigma(\mathscr{D}) \text{ is even;} \\ \frac{1}{2}d_r & \text{if } \sigma(\mathscr{D}) \text{ is odd,} \end{cases}$$

where $\sigma(\mathscr{D})$ is the sum of the elements in $\mathscr{D}$, $r = \max\{i : d_i \text{ is odd}\}$, and $C^\star$ is as defined in Proposition 1.

Proof. This follows from Theorem 4 when each d_i is even. Henceforth, we assume that at least one d_i is odd. As in Theorem 4, we show that there is an optimal degree sequence of the form given in Proposition 1. The result then follows.

Consider the sequence $\bar{\mathbf{s}}$ corresponding to $C = C^\star$, as defined in Proposition 1. It is easily verified that the sum of the elements of $\bar{\mathbf{s}}$ is given by the expressions for $\ell_q(\mathscr{D})$ in the appropriate cases. Since $\bar{\mathbf{s}}$ is graphic, this proves the upper bound on $\ell_q(\mathscr{D})$.

We prove that any graphic sequence with least element 2 and degree set $\mathscr{D}$ has size which is at least $\sigma(\bar{\mathbf{s}})$, proving the lower bound on $\ell_q(\mathscr{D})$. Let $\mathbf{s} = (d_1)_{m_1}, \ldots, (d_{n-1})_{m_{n-1}}, (2)_{m_n}$ be any graphic sequence with degree set $\mathscr{D}$. Repeatedly apply the Splitting lemma to replace all but one copies of even d_i by 2's and all but one copies of odd d_i by one d_r and 2's. This gives us a graphic sequence $\mathbf{u}$ with least element 2, degree set $\mathscr{D}$, with single copies of each d_i except d_r and 2.

Let G be a graph with degree sequence $\mathbf{u}$. Assume that G has more than two vertices of degree d_r, and let x and y be two of those vertices. Let $v_1, \ldots, v_{d_r}$ and $w_1, \ldots, w_{d_r}$ denote the neighbours of x and y respectively. Without loss of generality, assume $v_1 \neq w_1$ since $d_r > 1$. Construct the graph G' with vertex set $\big(V(G) \setminus \{x, y\}\big) \cup \{a_1, \ldots, a_\alpha\} \cup \{b_1, \ldots, b_\alpha\} \cup \{z\}$, where $\alpha = \frac{1}{2}(d_r - 1)$, and edge set created by removing from the edges of G those with endpoints x or y, adding the edges zv_1 and zw_1, the edges a_iv_{2i} and a_iv_{2i+1}, the edges b_iw_{2i} and b_iw_{2i+1} for $i \in \{1, \ldots, \frac{1}{2}(d_r - 1)\}$. Thus,

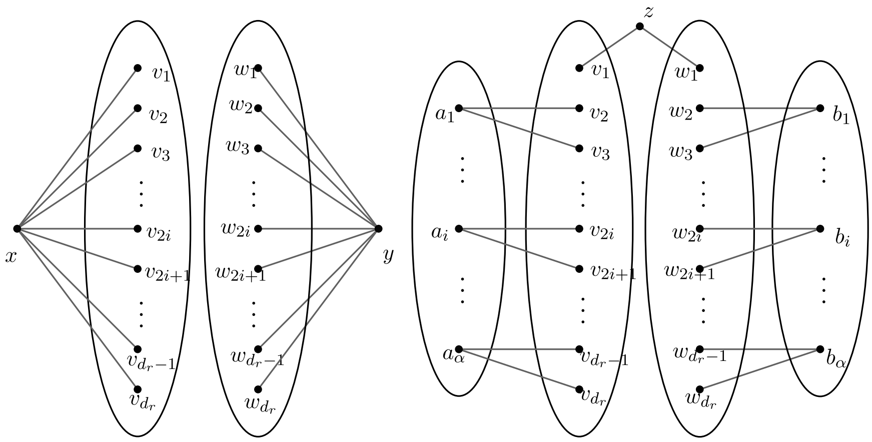


Fig. 2. Original graph G (left) and modified graph G' (right) from the proof of Theorem 5.

$$E(G') = \Big(E(G) \setminus \big(\{xv_i : 1 \le i \le d_r\} \bigcup \{yw_i : 1 \le i \le d_r\}\big)\Big) \bigcup \{zv_1, zw_1\}$$
$$\bigcup \{a_i v_{2i}, a_i v_{2i+1} : 1 \le i \le \tfrac{1}{2}(d_r - 1)\} \bigcup \{b_i w_{2i}, b_i w_{2i+1} : 1 \le i \le \tfrac{1}{2}(d_r - 1)\}.$$

Note that G' has degree set $\mathscr{D}$, with single copies of each d_i except of $d_n = 2$ and d_r, with the number of copies of d_r decreasing by two, and that G and G' have the same size. Repeated applications of this process results in a graph $\tilde{G}$ with degree set $\mathscr{D}$, with single copies of each d_i except of $d_n = 2$ and d_r, with the number of copies of d_r equal to 1 or 2, and that G and $\tilde{G}$ have the same size. The number of copies of d_r is determined by the parity of $\sigma(\mathscr{D})$. From Proposition 1, it follows that $\tilde{G}$, and consequently G, has size at least given by the expressions for $\ell_q(\mathscr{D})$ in the appropriate cases (see Fig. 2). ■

5. The case where $\mathscr{D}$ is an interval

By an interval we mean a set of the type $\{m, m+1, \ldots, n\}$, where m, n are positive integers with $m \le n$. We denote this by $[m, n]$. In this section, we first determine $\ell_q([1, n])$ using Corollary 1, and then use this to determine $\ell_q([m, n])$ when $m(m+1) < 2\lceil \frac{n}{2} \rceil$. The case where $m(m+1) \ge 2\lceil \frac{n}{2} \rceil$ is handled separately.

Theorem 6. *For $n \ge 1$,*

$$\ell_q\big([1, n]\big) = \frac{1}{2}\left(\frac{1}{2}n(n+1) + \left\lceil \frac{n}{2} \right\rceil\right).$$

Proof. We first determine $\Delta_{\mathbf{s}}(k)$ for the sequence $\mathbf{s} = n, n-1, \ldots, 1$ and $k \in \{1, \ldots, n\}$. If $k \ge \lceil \frac{n}{2} \rceil$, then $\min\{k, n-i\} = n - i$ for all $i \ge k+1$. Therefore,

$$\begin{aligned}\Delta_{\mathbf{s}}(k) &= k(k-1) + \sum_{i=k+1}^{n} \min\{k, n-i+1\} - \sum_{i=1}^{k}(n-i+1) \\ &= k(k-1) + \frac{(n-k)(n-k+1)}{2} - \frac{k(2n-k+1)}{2} \\ &= 2\left(k - \frac{n+1}{2}\right)^2 - \frac{n+1}{2}.\end{aligned}$$

If $k < \lceil \frac{n}{2} \rceil$, then $\sum_{i=k+1}^{n} \min\{k, n-i+1\} = \sum_{i=k+1}^{n-k} k + \sum_{i=n-k+1}^{n}(n-i+1) = k(n-2k) + \frac{k(k+1)}{2}$. Therefore, in this case, $\Delta_{\mathbf{s}}(k) = k(k-1) + k(n-2k) + \frac{k(k+1)}{2} - \frac{k(2n-k+1)}{2} = -k$. That is,

$$\Delta_{\mathbf{s}}(k) = \begin{cases} -k & \text{if } k < \lceil \frac{n}{2} \rceil; \\ 2\left(k - \frac{n+1}{2}\right)^2 - \frac{n+1}{2} & \text{if } k \ge \lceil \frac{n}{2} \rceil. \end{cases}$$

Consequently,

$$\begin{aligned}\max_{1\le k\le n}\{-\Delta_{\mathbf{s}}(k)\} &= \max\left\{\max_{1\le k<\lceil n/2\rceil}\{k\},\ \max_{\lceil n/2\rceil\le k\le n}\left\{\frac{n+1}{2}-2\left(k-\frac{n+1}{2}\right)^2\right\}\right\}\\ &= \max\left\{\left\lceil\frac{n}{2}\right\rceil-1,\ \frac{n+1}{2}-2\left(\left\lceil\frac{n}{2}\right\rceil-\frac{n+1}{2}\right)^2\right\}\le\left\lceil\frac{n}{2}\right\rceil.\end{aligned}$$

The result now follows from Corollary 1. ■

Proposition 2. *For any $m \ge 1$, $\ell_q\big([m,n]\big) \ge \ell_q\big([1,n]\big)$.*

Proof. Let G be any graph of size $\ell_q\big([m,n]\big)$ with degree set $[m,n]$. We use the Splitting lemma to construct a graph G' with degree set $[1,n]$, and having the same size as G, thereby proving the result.

Choose vertices $v_m,\dots,v_n$ in G such that $\deg v_i = i$ for each $i \in \{m,\dots,n\}$, and let $S = \{v_m,\dots,v_n\}$. Note that $|V(G)| \ge n+1$, so that $|V(G)\setminus S| \ge m$. Choose any $m-1$ vertices $w_1,\dots,w_{m-1}$ in $V(G)\setminus S$, and use the Splitting lemma to bifurcate each w_i into u_i and v_i such that $\deg v_i = i$ and $\deg u_i = \deg w_i - i$. Then the graph G' with $V(G') = \big(V(G)\setminus\{w_1,\dots,w_{m-1}\}\big)\cup\{u_1,\dots,u_{m-1}\}\cup\{v_1,\dots,v_{m-1}\}$ has the same size as G, and has degree set $[1,n]$. ■

Theorem 7. *Let m,n be positive integers such that $m(m+1) < 2\lceil\frac{n}{2}\rceil$. Then*

$$\ell_q\big([m,n]\big)=\ell_q\big([1,n]\big)=\frac{1}{2}\left(\frac{1}{2}n(n+1)+\left\lceil\frac{n}{2}\right\rceil\right).$$

Proof. We first construct a graph G with $\mathscr{D}(G) = [1,n]$ such that the size of G equals $\ell_q\big([1,n]\big)$.[2] Define $V(G) = X\cup Y\cup W$, where

$$X=\{x_1,\dots,x_{\lceil\frac{n}{2}\rceil-1}\},\quad Y=\{y_1,\dots,y_{\lfloor\frac{n}{2}\rfloor}\},\quad W=\{w,w^\star\}.$$

Define $E(G)$ by

$$\left\{x_iy_j : 1\le i\le\left\lceil\frac{n}{2}\right\rceil-1,\ 1\le j\le i\right\}\bigcup\left\{y_iy_j : 1\le i<j\le\left\lfloor\frac{n}{2}\right\rfloor\right\}\bigcup\{yz : y\in Y, z\in W\}\bigcup E_0(G),$$

where $E_0(G)$ is empty when n is even, and $E_0(G) = \{ww^\star\}$ when n is odd.

It is easy to verify that $\deg x_i = i$, $1 \le i \le \lceil\frac{n}{2}\rceil - 1$, $\deg y_i = n-i+1$, $1 \le i \le \lfloor\frac{n}{2}\rfloor$, and $\deg w = \deg w^\star = \lceil\frac{n}{2}\rceil$. Thus, G is a graph with degree set $[1,n]$ and size $\ell_q\big([1,n]\big) = \frac{1}{2}\left(\frac{1}{2}n(n+1)+\lceil\frac{n}{2}\rceil\right)$.

We now construct a graph G' with degree set $[m,n]$ and with size equal to the size of G. The only vertices in G that have degree less than m are $x_1,\dots,x_{m-1}$; in fact, $\deg x_i = i$ for $1\le i\le m-1$. We add $m-i$ edges to the vertex x_i for $i\in\{1,\dots,m-1\}$ and remove $1+2+3+\cdots+(m-1) = \frac{1}{2}m(m-1)$ edges from the vertex $w^\star$ without affecting the degrees of vertices in Y. This results in $\deg x_i = m$ for $i\in\{1,\dots,m-1\}$, $\deg w^\star = \lceil\frac{n}{2}\rceil - \frac{1}{2}m(m-1)+1$, with degrees of all other vertices in G' unchanged. Since $\lceil\frac{n}{2}\rceil > \frac{m(m+1)}{2}$, we have that $\deg w^\star > m+1 \ge m$ and $\deg w^\star \le n$.

Let $X' = \{x_1,\dots,x_{m-1}\}$ and $Y' = \left\{y_{\lfloor\frac{n}{2}\rfloor-\frac{m(m-1)}{2}+1},\dots,y_{\lfloor\frac{n}{2}\rfloor}\right\}$. Note that $|X'| = m-1$, $|Y'| = \frac{1}{2}m(m-1)$, and $x_i \not\leftrightarrow y_j$ for $x_i\in X'$ and $y_j\in Y'$. The nonadjacency in G between vertices of X' and Y' is a consequence of $\min j = \lfloor\frac{n}{2}\rfloor - \frac{1}{2}m(m-1)+1 > m-1 = \max i$. Note that the degrees of the vertices in Y' occupy the $\frac{1}{2}m(m-1)$ consecutive integers starting with $\lceil\frac{n}{2}\rceil+1$.

Partition Y' into sets $Y'_1,\dots,Y'_{m-1}$ such that $|Y'_i| = m-i$, $1\le i\le m-1$. To construct G' from G, remove the $\frac{1}{2}m(m-1)$ edges $w^\star y_j$, $y_j\in Y'$, and join x_i to each vertex in Y'_i, for $i\in\{1,\dots,m-1\}$. This construction is illustrated in Fig. 3.

It is clear that G' has the same size as G, the degree of vertices in X' are all equal to m, and the degrees of vertices in Y' are unchanged, and $\deg w^\star = \lceil\frac{n}{2}\rceil - \frac{1}{2}m(m-1)+1 \in [m,n]$. Thus G' is a graph with desired properties.

We note that G and G' have size $\ell_q\big([1,n]\big)$ and $\mathscr{D}(G') = [m,n]$. We further note that $\ell_q\big([1,n]\big)$ provides a lower bound for $\ell_q\big([m,n]\big)$ by Proposition 2. This completes the proof. ■

The sequence among

$$\mathbf{s}_1 = n, n-1,\dots,m+1,(m)_{m+1},\qquad \mathbf{s}_2 = n, n-1,\dots,(m+1)_2,(m)_m \tag{3}$$

with even sum must have optimum size $\ell_q([m,n])$ *provided* it is graphic. We prove this is the case when $m(m+1)\ge 2\lceil\frac{n}{2}\rceil$.

[2] We remark that this graph is defined inductively in [Lemma 2, [13]].

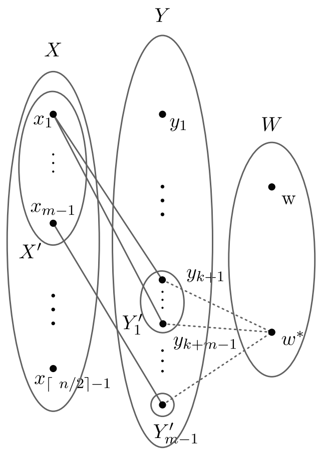


Fig. 3. Edge differences between graphs G and G' in the proof of Theorem 7. Edges added to graph G' are in solid, edges removed from G are dotted, and unchanged edges are omitted for clarity.

Theorem 8. *Let m, n be positive integers such that $m \le n \le 2\lceil \frac{n}{2} \rceil \le m(m+1)$. Then exactly one of the sequences $\mathbf{s}_1$, $\mathbf{s}_2$ in eqn. (3) is graphic. In particular,*

$$\ell_q([m,n]) - \ell_q([1,n]) = \begin{cases} \frac{m(m-1)}{4} & \text{if } m \equiv 0, 1 \pmod 4; \\ \frac{m(m-1)+2}{4} & \text{if } m \equiv 2, 3 \pmod 4). \end{cases}$$

Proof. When $m > \lceil \frac{n}{2} \rceil$, then $\min \mathscr{D} = m \ge n - m + 1 = |\mathscr{D}|$, and therefore the result is a consequence of [13, Theorem 4]. Henceforth, we assume $m \le \lceil \frac{n}{2} \rceil$.

Observe that the sum of the integers in $\mathbf{s}_1$ and $\mathbf{s}_2$ differ by one. Hence, *at most* one of the two sequence can be graphic. We use Theorem 3 to prove that the sequence with even sum is a graphic sequence. In particular, we show that both sequences satisfy the inequalities in Theorem 3.

Note that

$$d_i = \begin{cases} n - i + 1 & \text{if } 1 \le i \le n - m; \\ m & \text{if } n - m + 1 \le i \le n + 1. \end{cases}$$

for sequence $\mathbf{s}_1$, and

$$d_i = \begin{cases} n - i + 1 & \text{if } 1 \le i \le n - m - 1; \\ m + 1 & \text{if } n - m \le i \le n - m + 1; \\ m & \text{if } n - m + 2 \le i \le n + 1. \end{cases}$$

for sequence $\mathbf{s}_2$.

Since $m + 1 = d_{n-m} \ge n - m - 1$ for the sequence $\mathbf{s}_1$ and $m + 1 = d_{n-m+1} \ge n - m$ for the sequence $\mathbf{s}_2$, while $m = d_{n+1} \le n + 1$ for both sequences, the inequality in Theorem 3 needs to be checked only for $1 \le k \le t$, where $t = n - m$ for the sequence $\mathbf{s}_1$ and $t = n - m + 1$ for the sequence $\mathbf{s}_2$.

Thus, we must show that

$$\Delta_{\mathbf{s}}(k) = k(k-1) + \sum_{i=k+1}^{n+1} \min\{k, d_i\} - \sum_{i=1}^{k} d_i \ge 0 \quad \text{for} \quad 1 \le k \le t$$

for each of the sequences.

For the sequence $\mathbf{s}_1$ and $1 \le k \le m$,

$$\begin{aligned} \Delta_{\mathbf{s}_1}(k) &= k(k-1) + \sum_{i=k+1}^{n-m} \min\{k, n-i+1\} + \sum_{i=n-m+1}^{n+1} \min\{k, m\} - \sum_{i=1}^{k} (n-i+1) \\ &= k(k-1) + (n-m-k)k + (m+1)k - \frac{k(2n-k+1)}{2} \\ &= \frac{1}{2}k(k-1) \\ &\ge 0. \end{aligned}$$

When $m < k \le \lceil \frac{n}{2} \rceil$,

$$\begin{aligned}
\Delta_{\mathbf{s}_1}(k) &= k(k-1) + \sum_{i=k+1}^{n-k} \min\{k, n-i+1\} + \sum_{i=n-k+1}^{n-m} \min\{k, n-i+1\} + \sum_{i=n-m+1}^{n+1} \min\{k, m\} \\
&\quad - \sum_{i=1}^{k} (n-i+1) \\
&= k(k-1) + (n-2k)k + (k + \cdots + (m+1)) + (m+1)m - \frac{1}{2}k(2n-k+1) \\
&= -k + \frac{1}{2}m(m+1) \\
&\ge -\left\lceil \frac{n}{2} \right\rceil + \frac{1}{2}m(m+1) \\
&\ge 0,
\end{aligned}$$

where the inequalities hold since $k \le \lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} \rceil \le \frac{1}{2}m(m+1)$.

The case $\lceil \frac{n}{2} \rceil < k \le m$ is vacuous since in this case $t = n - m + 1 \le \lceil \frac{n}{2} \rceil < k$. Finally, when $\lceil \frac{n}{2} \rceil < k \le t$,

$$\begin{aligned}
\Delta_{\mathbf{s}_1}(k) &= k(k-1) + \sum_{i=k+1}^{n-m} \min\{k, n-i+1\} + \sum_{i=n-m}^{n+1} \min\{k, m\} - \sum_{i=1}^{k} (n-i+1) \\
&= k(k-1) + \sum_{i=k+1}^{n-m} (n-i+1) + (m+1)m - \frac{1}{2}k(2n-k+1) \\
&= 2k(k-1) + \frac{1}{2}n(n+1) - 2nk + \frac{1}{2}m(m+1) \\
&= 2\left(k - \frac{n+1}{2}\right)^2 - \frac{n+1}{2} + \frac{1}{2}m(m+1) \\
&\ge -\left\lceil \frac{n}{2} \right\rceil + \frac{1}{2}m(m+1) \\
&\ge 0.
\end{aligned}$$

The calculations for the sequence $\mathbf{s}_2$ are similar, and omitted here. ■

6. An approximation solution for the general case

Now, given any set $\mathscr{D}$ of positive integers, we construct a graph with degree set $\mathscr{D}$ whose size differs from $\ell_q(\mathscr{D})$ by at most $(\min \mathscr{D} - 1)$. In particular, we achieve an optimal graph in the case where $\min \mathscr{D} = 1$. We show that the sequence constructed in Proposition 1 achieves this.

Lemma 2. *Let $\mathscr{D} = \{d_1, \ldots, d_n\}$ be a set of positive integers arranged in decreasing order. Let $a_1, \ldots, a_p$ denote the graphic sequence $\bar{\mathbf{s}}$ with degree set $\mathscr{D}$, corresponding to $C = C^\star$, as given in Proposition 1. If $a'_1, \ldots, a'_{p'}$ denotes any graphic sequence with degree set $\mathscr{D}$, then $a_i \le a'_i$ for $1 \le i \le \min\{p, p'\}$.*

Proof. The lemma is clear for the sequence $\bar{\mathbf{s}}$ in Proposition 1(a), that is, when $\sigma(\mathscr{D})$ is even or d_n is odd.

Now assume that $\sigma(\mathscr{D})$ is odd and d_n is even, so that the sequence $\bar{\mathbf{s}}$ is as defined in Proposition 1(b). We have $a_i = d_i \le a'_i$ for $1 \le i \le r$. Since $\sigma(\mathscr{D})$ is odd, at least one odd d_i, say d_j, must be repeated. Since $d_j \ge d_r$, we have $a'_i \ge d_{i-1} = a_i$ for $n+1 \ge i \ge r$. The inequality is clear for $i > n+1$. ■

Theorem 9. *Let $\mathscr{D} = \{d_1, \ldots, d_n\}$ be a set of positive integers arranged in decreasing order. Then for the graphic sequence $\bar{\mathbf{s}}$ constructed in Proposition 1 we have*

$$\frac{1}{2}\sigma\left(\bar{\mathbf{s}}\right) - \ell_q(\mathscr{D}) < \min \mathscr{D}.$$

Proof. Let the sequence $\bar{\mathbf{s}} = a_1, \ldots, a_p$, as constructed in Proposition 1. Let $\bar{\mathbf{s}}' = a'_1, a'_2, \ldots, a'_{p'}$ be any degree sequence for a graph G with degree set $\mathscr{D}$ and $\ell_q(\mathscr{D})$ edges. Let $\sigma\left(\bar{\mathbf{s}}'\right) = 2\ell_q(\mathscr{D}) = \sum_{i=1}^{p'} a'_i$.

Thus, we only need to show the following for the result to hold:

$$\sigma\left(\bar{\mathbf{s}}\right) < \sigma\left(\bar{\mathbf{s}}'\right) + 2d_n. \tag{4}$$

From Lemma 2, we see that

$$a_i \leq a'_i \qquad \forall\ 1 \leq i \leq \min\{p, p'\} \tag{5}$$

Recall that $k^\star$ is defined in Proposition 1(a), and can be modified suitably for Proposition 1(b). We use this notation in either case. Let ℓ' be the largest index such that $a'_{\ell'} \geq k^\star$ and $a'_{\ell'+1} < k^\star$ and ℓ be the largest index such that $a_\ell \geq k^\star$ and $a_{\ell+1} < k^\star$.

From inequations (5), it follows that

$$\ell \leq \ell'. \tag{6}$$

Therefore

$$\sigma\left(\bar{\mathbf{s}}\right) = \sum_{i=1}^{p} a_i = \sum_{i=1}^{k^\star} a_i + \sum_{i=k^\star+1}^{p} a_i = \sum_{i=1}^{k^\star} a_i + \sum_{i=k^\star+1}^{\ell} \left(a_i - k^\star\right) + \sum_{i=k^\star+1}^{p} \min(k^\star, a_i).$$

Let $\sigma_1 = \sum_{i=1}^{k^\star} a_i$, $\sigma_2 = \sum_{i=k^\star+1}^{\ell}\left(a_i - k^\star\right)$, and $\sigma_3 = \sum_{i=k^\star+1}^{p} \min(k^\star, a_i)$.

Similarly,

$$\sigma\left(\bar{\mathbf{s}}'\right) = \sum_{i=1}^{k^\star} a'_i + \sum_{i=k^\star+1}^{\ell'} \left(a'_i - k^\star\right) + \sum_{i=k^\star+1}^{p'} \min(k^\star, a'_i).$$

Let $\sigma'_1 = \sum_{i=1}^{k^\star} a'_i$, $\sigma'_2 = \sum_{i=k^\star+1}^{\ell'}\left(a'_i - k^\star\right)$, and $\sigma'_3 = \sum_{i=k^\star+1}^{p'} \min(k^\star, a'_i)$.

We will show the following three inequalities hold:

$$\sigma_1 \leq \sigma'_1, \quad \sigma_2 \leq \sigma'_2, \quad \sigma_3 < \sigma'_3 + 2d_n \tag{7}$$

These imply inequations (4).

From inequations (5) and inequations (6), we get $\sigma_1 \leq \sigma'_1$ and $\sigma_2 \leq \sigma'_2$.

To show that $\sigma_3 \leq \sigma'_3 + 2d_n$, we apply the Erdős–Gallai condition to the graphic sequences $\bar{\mathbf{s}}$ and $\bar{\mathbf{s}}'$ at $k^\star$, and use Proposition 1 to bound $\Delta_{\bar{\mathbf{s}}}(k^\star)$:

$$0 \leq \Delta_{\bar{\mathbf{s}}}(k^\star) = \sigma_3 - \sigma_1 + k^\star(k^\star - 1) < 2d_n, \qquad 0 \leq \Delta_{\bar{\mathbf{s}}'}(k^\star) \leq \sigma'_3 - \sigma'_1 + k^\star(k^\star - 1).$$

From inequations (7) and the above inequations, we now have

$$\sigma_3 < \sigma_1 - k^\star(k^\star - 1) + 2d_n \leq \sigma'_1 - k^\star(k^\star - 1) + 2d_n \leq \sigma'_3 + 2d_n. \quad \square$$

Remark 1. The sequence $\bar{s}$ in Theorem 9 is a $(1 + M)$-approximation solution to the problem, where $M = \min\left\{\frac{2}{d_1+1}, \frac{2(\sqrt{2}-1)}{n-1}\right\}$.

Proof. Combining the result of Theorem 9 with the basic inequality $2\ell_q(\mathscr{D}) > (d_1 + 1)d_n$ yields

$$\frac{\sigma\left(\bar{\mathbf{s}}\right)}{2\ell_q(\mathscr{D})} < 1 + \frac{d_n}{\ell_q(\mathscr{D})} < 1 + \frac{2}{d_1 + 1}.$$

This results in the first bound.

To obtain the second bound, we use a sharper lower bound $2\ell_q(\mathscr{D}) \geq \left(\sum_{i=1}^{n} d_i\right) + (d_1 + 1 - n)d_n$ to get

$$\begin{aligned}
2\ell_q(\mathscr{D}) &\geq \left(\sum_{i=1}^{n} d_i\right) + (d_1 + 1 - n)d_n \\
&\geq \left(\sum_{i=1}^{n} d_n + (n - i)\right) + \left(d_1 - (n - 1)\right)d_n \\
&\geq nd_n + \frac{1}{2}n(n-1) + d_n^2 \\
&\geq d_n\left(n + \frac{n(n-1)}{2d_n} + d_n\right) \\
&\geq d_n\left(n + 2\sqrt{\frac{n(n-1)}{2}}\right)
\end{aligned}$$

$$> d_n \left((n-1) + 2\sqrt{\frac{(n-1)^2}{2}} \right)$$
$$= (1+\sqrt{2})(n-1)d_n. \tag{8}$$

From Theorem 9, we have $\frac{\sigma(\bar{\mathbf{s}})}{2\ell_q(\mathscr{D})} \leq 1 + \frac{d_n}{2\ell_q(\mathscr{D})}$. Combining this with inequation (8) yields

$$\frac{\sigma(\bar{\mathbf{s}})}{2\ell_q(\mathscr{D})} < 1 + \frac{d_n}{\ell_q(\mathscr{D})} < 1 + \frac{2(\sqrt{2}-1)}{n-1}.$$

This proves the second bound and hence our claim. ■

Since our result shows that $2\ell_q(\mathscr{D}) \in \left(\sigma(\bar{\mathbf{s}}) - 2\min \mathscr{D}, \sigma(\bar{\mathbf{s}})\right]$, a natural algorithm to determine $\ell_q(\mathscr{D})$ would be to perform a search over this interval. Thus, for each even σ in the interval, we wish to determine if there is a graph G with $\sigma/2$ edges such that $\mathscr{D}(G) = \mathscr{D}$. One way to do this is to determine all solutions to $\sigma = \sum_i m_i d_i$ in positive integers m_i, and then check whether any of the corresponding sequences is graphic. This latter problem can be solved in polynomial time in $\sum_i m_i > d_1$ for fixed $m_1, \ldots, m_n$, which is exponential in the input; refer [2,3,11]. The former problem is the well-known Frobenius Coin problem, which has a rich and long history, with several applications and extensions, and connections to several areas of research. The Frobenius Coin problem can be solved in polynomial-time in $d_1 \sum_i m_i$, and is known to be NP-hard [4,10].

Therefore, given a σ, we must (i) determine the existence of positive integers $m_1, \ldots, m_n$ such that $\sigma = \sum_i m_i d_i$, and (ii) determine whether $(d_1)_{m_1}, \ldots, (d_n)_{m_n}$ is graphic for each solution $m_1, \ldots, m_n$ in (i). Both these problems have no known polynomial-time algorithms in our input size $\sum_i \log d_i$. While our problem imposes more structure than each of the those problems, we speculate that it is also NP-hard, in which case our result in Theorem 9 assumes larger significance.

Data availability

No data was used for the research described in the article.

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