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On deletions from numerical semigroups generated by modified arithmetic progressions that leave the Frobenius number invariant

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Abstract
For a numerical semigroup S, we denote by $\mathrm{F}(S)$ the Frobenius number of S. For numerical semigroups generated by modified arithmetic sequences A, we characterize (i) $a \in A$ for which $\mathrm{F}(\langle A \setminus \{a\}\rangle) = \mathrm{F}(\langle A\rangle)$, and (ii) characterize 3-subsets A' of A for which $\mathrm{F}(\langle A'\rangle) = \mathrm{F}(\langle A\rangle)$, and explore some 4-subsets and some subsets of cardinality roughly $\sqrt{k}$ for some choices of k for which the Frobenius number is the same as that of A.

Keywords Numerical semigroup · Embedding dimension · Apéry set · Frobenius number

Mathematics Subject Classification 11D07 · 11D04

1 Introduction

A numerical semigroup S is a submonoid of $\mathbb{Z}_{\geq 0}$ that has a finite complement $\mathrm{G}(S)$ in $\mathbb{Z}_{\geq 0}$. Given a set $A = \{a_1, \ldots, a_n\}$ of positive integers with $\gcd(a_1, \ldots, a_n) = 1$, let

$$\langle A\rangle = \{a_1x_1 + \cdots + a_kx_n : x_i \geq 0\}$$

denote the numerical semigroup generated by A. Every numerical semigroup S is of the form $\langle A\rangle$ for some finite set A. Moreover, every S has a unique minimal generating set, the cardinality of which is called the embedding dimension of the semigroup and denoted by $\mathbf{e}(S)$.

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The Frobenius number of S is defined as the largest element in $\mathrm{G}(S)$:

$$\mathrm{F}(S) = \max \mathrm{G}(S).$$

A very useful tool in the study of numerical semigroups is the determination of Apéry sets of the semigroup. Given a numerical semigroup S generated by A and $a \in S \setminus \{0\}$, the Apéry set $\mathrm{Ap}(S, a)$ of S corresponding to a is given by

$$\mathrm{Ap}\,(S, a) = \{\mathbf{m}_x : 0 \leq x \leq a - 1\},$$

where $\mathbf{m}_x$ denotes the least positive integer in S congruent to x modulo a. The following result, due to Brauer and Shockley [3], shows how $\mathrm{F}(S)$ can be determined from the Apéry set of S corresponding to any $a \in S \setminus \{0\}$.

Lemma 1.1 (Brauer and Shockley [3])
Let S be the numerical semigroup generated by A, and let $a \in A$. Then

$$F(S) = \left(\max_{1 \leq x \leq a-1} \boldsymbol{m}_x \right) - a.$$

Determining the Frobenius number is a difficult problem in general. Two of the special cases where $\mathrm{F}(S)$ has been exactly determined are when:

(i) $A = \mathrm{AP}(a, d, k)$ is the k-term arithmetic sequence with first term a and common difference d, with $\gcd(a, d) = 1$; see [1, 9, 13], and
(ii) $A = \mathrm{GP}(a, b, k)$ is the k-term geometric sequence with first term a^{k-1} and common ratio b/a, with $\gcd(a, b) = 1$; see [6, 14].

Rødseth [10] studied the problem of determining single-element extensions of $A = \mathrm{AP}(a, d, k)$ which leaves the Frobenius number of the numerical semigroup generated by A unchanged. Further studies on this problem, including a characterization of single-element extensions and extensions of maximum cardinality were carried out by Ritter [8], and by Batra, Kumar and Tripathi [2].

Ritter [7] studied the analogous problem of removal of elements from modified arithmetic sequences which leave the Frobenius number of the corresponding numerical semigroup unchanged. More specifically, he found bounds on the function that counts the largest number of deletions from a modified arithmetic sequence while leaving the Frobenius number unchanged. He also determined extremal sets that achieve these bounds under certain restrictions. More recently, Elizeche and Tripathi [4] studied the problem of removal of elements from compound sequences (which are generalizations of geometric sequences) which leave the Frobenius number of the corresponding numerical semigroup unchanged.

In this paper, for modified arithmetic sequences A, we explore the following two problems:

(i) Characterize $a \in A$ for which $\mathrm{F}(\langle A \setminus \{a\}\rangle) = \mathrm{F}(\langle A\rangle)$; see Theorem 3.5.

(ii) Characterize 3-subsets A' of A for which $\mathrm{F}(\langle A'\rangle) = \mathrm{F}(\langle A\rangle)$; see Theorem 4.2. Further, we explore some 4-subsets and some subsets of cardinality roughly $\sqrt{k}$ for some choices of k for which the Frobenius number is the same as that of A; see Theorems 4.4, 4.6, 4.7, 4.8.

2 Preliminary results

The Frobenius number for arithmetic sequences was first determined by Roberts [9]; see also Bateman [1] and Tripathi [13]. Modified arithmetic sequences generalize arithmetic sequences:

$$\mathrm{MAP}(a, d, h, k) = \{a, ha + d, ha + 2d, \ldots, ha + (k-1)d\},$$

with $\gcd(a, d) = 1$, $h \geq 1$ and $k \geq 2$. It is no loss of generality to assume $k \leq a$, since each of the elements of the form $ha + kd$ with $k \geq a$ belongs to $\langle \mathrm{MAP}(a, d, h, a)\rangle$. Observe that the case $h = 1$ gives the arithmetic sequence $\mathrm{AP}(a, d, k)$. Selmer [11] determined the Frobenius number for modified arithmetic sequences; see also Tripathi [12].

Throughout this paper, let $A = \mathrm{MAP}(a, d, h, k)$, where a, d, h, k are positive integers with $\gcd(a, d) = 1$ and $3 \leq k \leq a$. In Sect. 3, we characterize elements $x \in A$ for which $\mathrm{F}(\langle A \setminus \{x\}\rangle) = \mathrm{F}(\langle A\rangle)$. In Sect. 4, we characterize subsets B of A of least cardinality for which $\mathrm{F}(\langle B\rangle) = \mathrm{F}(\langle A\rangle)$. Therefore, each such B is obtained from A by a maximum removal of elements of A.

Lemma 2.1 (Tripathi, [12, Lemma 3])
Let a, d, h, k be positive integers, where $\gcd(a, d) = 1$ and $k \geq 3$. Let $A = MAP(a, d, h, k)$. For $x \in \{1, \ldots, a-1\}$, the least integer in $\langle A\rangle$ that is congruent to dx modulo a is given by

$$\boldsymbol{m}_{dx} = ha\left(1 + \left\lfloor \tfrac{x-1}{k-1} \right\rfloor\right) + dx.$$

From Lemma 1.1 and Lemma 2.1 it easily follows that

$$\mathrm{F}(\langle A\rangle) = ha\left\lfloor \tfrac{a-2}{k-1} \right\rfloor + d(a-1) + (h-1)a. \tag{1}$$

3 Deletion of a single element

Let a, d, h, k be positive integers with $\gcd(a, d) = 1$, $h \geq 1$ and $2 \leq k \leq a$, and let $A = \mathrm{MAP}(a, d, h, k)$. In this Section, we characterize $x \in A$ for which the numerical semigroup generated by $A \setminus \{x\}$ has Frobenius number equal to $\mathrm{F}(\langle A\rangle)$. We denote the set of all such elements by A^-. In Proposition 3.2, we show that a and $ha + d$ in A are never in A^-, and give conditions under which $ha + (k-1)d$ is in A^-. In Theorem 3.3 and Theorem 3.4, we determine all such elements $x \in A$ in the cases $3 < k < a$ and

$k = a$, respectively. Finally, we summarize the results from this Section in Theorem 3.5.

Definition 3.1 Let a, d, h, k be positive integers, where $\gcd(a, d) = 1$ and $3 \leq k \leq a$. For $A = \mathrm{MAP}(a, d, h, k)$, define

$$A^- = \{ha + jd : \mathrm{F}(\langle A \setminus \{ha + jd\}\rangle) = \mathrm{F}(\langle A\rangle), 1 \leq j \leq k-1\} \bigcup \{a : \mathrm{F}(\langle A \setminus \{a\}\rangle) = \mathrm{F}(\langle A\rangle)\}.$$

Proposition 3.2 *Let a, d, h, k be positive integers, where $a \geq 3$, $\gcd(a, d) = 1$ and $3 \leq k \leq a$. Let $A = MAP(a, d, h, k)$. Then*

(i) $a \notin A^-$.
(ii) $ha + d \notin A^-$.
(iii) $ha + (k-1)d \in A^-$ *if and only if* $\left\lfloor \frac{a-2}{k-1} \right\rfloor = \left\lfloor \frac{a-2}{k-2} \right\rfloor$.

In particular, $A^- = \emptyset$ if $k = 3$.

Proof (i) Note that $A \setminus \{a\}$ is a numerical semigroup if and only if $\gcd(ha, d) = 1$. Assuming $\gcd(h, d) = 1$, we can write $A \setminus \{a\} = \mathrm{MAP}(ha + d, d, 1, k-1) = \mathrm{AP}(ha + d, d, k-1)$. By Eq. (1),

$$\begin{aligned} \mathrm{F}(\langle A \setminus \{a\}\rangle) &= (ha + d)\left\lfloor \tfrac{ha+d-2}{k-2} \right\rfloor + d(ha + d - 1) \\ &> ha\left\lfloor \tfrac{a-2}{k-2} \right\rfloor + d\left((a-1) + (h-1)a\right) \\ &> \mathrm{F}(\langle A\rangle). \end{aligned}$$

(iii) Since $A \setminus \{ha + (k-1)d\} = \mathrm{MAP}(a, d, h, k-1)$,

$$\begin{aligned} \mathrm{F}(\langle A \setminus \{ha + (k-1)d\}\rangle) &= ha\left\lfloor \tfrac{a-2}{k-2} \right\rfloor + d(a-1) + (h-1)a \\ &= \mathrm{F}(\langle A\rangle) \text{ if and only if} \\ &\left\lfloor \tfrac{a-2}{k-2} \right\rfloor = \left\lfloor \tfrac{a-2}{k-1} \right\rfloor \end{aligned}$$

by Eq. (1).

(ii) For the remainder of this proof, let $A^\star = A \setminus \{ha + d\}$. For $x \in \{1, \ldots, a-1\}$, let $\mathbf{m}^\star_{dx}$ denote the least positive integer in $\langle A^\star\rangle$ congruent to dx modulo a. We claim that $\mathbf{m}^\star_d > \mathbf{m}_{d(a-1)}$, so that $\mathrm{F}(\langle A^\star\rangle) > \mathrm{F}(\langle A\rangle)$ by Lemma 1.1, Lemma 2.1 and Eq. (1).
If $\mathbf{m}^\star_d = ma + d$, then there exist nonnegative integers $x_0, x_2, x_3, \ldots, x_{k-1}$ such that

$$ma + d = ax_0 + \sum_{i=2}^{k-1}(ha + id)x_i = a\left(x_0 + h\sum_{i=2}^{k-1} x_i\right) + d\sum_{i=2}^{k-1} i x_i. \qquad (2)$$

Reducing modulo a yields $\sum_{i=2}^{k-1} ix_i \equiv 1 \pmod{a}$ since $\gcd(a,d) = 1$. In order to minimize m in Eq. (2), we must set $x_0 = 0$. Thus, we must choose nonnegative integers $x_2, \ldots, x_{k-1}$ satisfying $\sum_{i=2}^{k-1} ix_i = at+1$ for some $t \geq 1$, and so $m = h\sum_{i=2}^{k-1} x_i + dt$ by Eq. (2). Hence the minimum value of m is

$$m_0 = \min_{t\geq 1}\left\{h\sum_{i=2}^{k-1} x_i + dt\right\} \text{ subject to } \sum_{i=2}^{k-1} ix_i = at+1. \tag{3}$$

Fix $t \geq 1$. In order to minimize m in Eq. (3), we must choose x_{k-1} as large as possible. If $at+1 = q(k-1)+r$, $0 \leq r < k-1$, the optimal choice of x_{k-1} is q. If $r = 0$, each other $x_i = 0$; and if $r > 1$, then $x_r = 1$ and all other $x_i = 0$. If $r = 1$, then $at = q(k-1)$, and the optimal choice of x_{k-1} is $q-1$. Now $\sum_{i=2}^{k-2} ix_i = k$, and an optimal choice is $x_2 = x_{k-2} = 1$ and all other $x_i = 0$. We conclude that the minimum value of $\sum_{i=2}^{k-1} x_i$ is q if $(k-1) \mid (at+1)$ and $q+1$ if $(k-1) \nmid (at+1)$. We may combine the two cases by stating that the minimum value of $\sum_{i=2}^{k-1} x_i$ is $\lfloor \frac{at}{k-1} \rfloor + 1$.
From Eq. (3) we now have

$$m_0 = \min_{t\geq 1}\left\{h\sum_{i=2}^{k-1} x_i + dt\right\} = \min_{t\geq 1}\left\{h\left(\left\lfloor \tfrac{at}{k-1}\right\rfloor + 1\right) + dt\right\} = h\left(\left\lfloor \tfrac{a}{k-1}\right\rfloor + 1\right) + d.$$

Therefore,

$$\mathbf{m}_d^\star = m_0 a + d = ha\left(\left\lfloor \tfrac{a}{k-1}\right\rfloor + 1\right) + d(a+1) > ha\left(\left\lfloor \tfrac{a-2}{k-1}\right\rfloor + 1\right) + d(a-1) = \mathbf{m}_{d(a-1)},$$

as claimed.

□

Theorem 3.3 *Let a, d, h, k be positive integers, where $a \geq 3$, $\gcd(a,d) = 1$ and $4 \leq k \leq a-1$. Let $A = MAP(a,d,h,k)$. Then*

$$A^- = \{ha + jd : 2 \leq j \leq k-3\} \bigcup S \bigcup T,$$

where

$$S = \begin{cases} \emptyset & \text{if } (k-1) \mid a \text{ or } (k-1) \mid (a-1), d < a; \\ \{ha + (k-2)d\} & \text{otherwise,} \end{cases}$$

and

$$T = \begin{cases} \{ha + (k-1)d\} & \text{if } \lfloor \frac{a-2}{k-1} \rfloor = \lfloor \frac{a-2}{k-2} \rfloor; \\ \emptyset & \text{if } \lfloor \frac{a-2}{k-1} \rfloor \neq \lfloor \frac{a-2}{k-2} \rfloor. \end{cases}$$

Proof We have $a, ha+d \notin A^-$ and $ha+(k-1)d \in A^-$ if and only if $\lfloor \frac{a-2}{k-1} \rfloor = \lfloor \frac{a-2}{k-2} \rfloor$, all by Proposition 3.2. There remains to consider whether or not $ha + jd \in A^-$ for $2 \leq j \leq k-2$.

Fix $j \in \{2, \ldots, k-2\}$, and let $A^\star = A \setminus \{ha + jd\}$. For $x \in \{1, \ldots, a-1\}$, let $\mathbf{m}^\star_{dx}$ denote the least positive integer in $\langle A^\star \rangle$ congruent to dx modulo a. We follow the general line of argument used in the proof of Proposition 3.2, part (ii).

If $\mathbf{m}^\star_{dx} = ma + dx$, then there exist nonnegative integers $x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k-1}$ such that

$$ma+dx = ax_0+\sum_{\substack{1\leq i\leq k-1\\ i\neq j}} (ha+id)x_i = a\left(x_0 + h\sum_{\substack{1\leq i\leq k-1\\ i\neq j}} x_i\right)+d\sum_{\substack{1\leq i\leq k-1\\ i\neq j}} ix_i. \quad (4)$$

Reducing modulo a, we have $\sum_{1\leq i\leq k-1, i\neq j} ix_i \equiv x \pmod a$. In order to minimize m in Eq. (4), we must set $x_0 = 0$, and hence we choose nonnegative integers $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k-1}$ satisfying $\sum_{1\leq i\leq k-1, i\neq j} ix_i = at + x$ for some $t \geq 0$. This implies $m = h\sum_{1\leq i\leq k-1, i\neq j} x_i + dt$ by Eq. (4). Hence the minimum value of m is

$$m_0 = \min_{t\geq 0}\left\{h\sum_{\substack{1\leq i\leq k-1\\ i\neq j}} x_i + dt\right\} \text{ subject to } \sum_{\substack{1\leq i\leq k-1\\ i\neq j}} ix_i = at + x. \quad (5)$$

Fix $t \geq 0$. Let $at + x = q(k-1) + r$, $0 \leq r < k-1$. We treat the cases $r \neq j$ and $r = j$ separately.

If $r \neq j$, then the optimal choice for x_{k-1} is q. If $r = 0$, choose each other $x_i = 0$; if $r > 0$, choose $x_r = 1$ and all other $x_i = 0$.

Suppose $r = j$. If $q = 0$, then an optimal choice is $x_1 = x_{j-1} = 1$ (in case $j = 2$, choose $x_1 = 2$) and all other $x_i = 0$. If $q > 0$ and $j < k-2$, then the optimal choice for x_{k-1} is $q-1$, $x_{j+1} = x_{k-2} = 1$ (in case $j = k-3$, choose $x_{k-2} = 2$) and all other $x_i = 0$. If $q > 0$ and $j = k-2$, then an optimal choice is $x_{k-1} = q$, $x_1 = x_{k-3} = 1$ and all other $x_i = 0$. Observe that in this case, the choice $x_{k-1} = q-1$ leaves us to minimize $\sum_{1\leq i\leq k-3} x_i$ subject to $\sum_{1\leq i\leq k-3} ix_i = 2k-3$, and this minimum is greater than 2 since $2(k-3) < 2k-3$.

We conclude that the minimum value of $\sum_{1\leq i\leq k-1, i\neq j} x_i$ is q if $(k-1) \mid (at+x)$ and $q+1$ otherwise, except that it equals 2 when $at + x = j$ and equals $q+2$ when $at + x = q(k-1) + k - 2$. This minimum is clearly achieved when $t = 0$, and we have

$$m_0 = \begin{cases} 2h & \text{if } x = j;\\ (q+2)h & \text{if } x = q(k-1)+k-2, q \geq 0;\\ \frac{x}{k-1}h & \text{if } (k-1) \mid x;\\ \left(\lfloor \frac{x}{k-1} \rfloor + 1\right)h & \text{otherwise.}\end{cases} \quad (6)$$

In view of the second case for m_0 above, we consider the cases $2 \leq j \leq k-3$ and $j = k-2$ separately.

CASE (i). If $2 \le j \le k-3$, then $m_0 = \left(\lfloor \frac{x-1}{k-1} \rfloor + 1\right) h$ except when $x = j$. Thus, $\mathbf{m}^\star_{dx} = \mathbf{m}_{dx}$ for all $x \ne j$, by Lemma 2.1. Since $\mathbf{m}^\star_{dj} = 2ha + dj < \mathbf{m}_{d(a-1)}$, $ha + jd \in A^-$ for all j in this case.

CASE (ii). If $j = k-2$, then $m_0 = \left(\lfloor \frac{x-1}{k-1} \rfloor + 1\right) h$ except when $x = j$ or $x = q(k-1)+k-2$, $q \ge 0$. Thus, $\mathbf{m}^\star_{dx} = \mathbf{m}_{dx}$ for all $x \ne j$ and $x \ne q(k-1)+k-2$, $q \ge 0$, by Lemma 2.1.

Let $\overline{a}$ be the largest integer of the form $q(k-1)+k-2$ that does not exceed $a-1$. Since $m^\star_{dj} < \mathbf{m}^\star_{d\overline{a}}$, we have

$$ha + jd \in A^- \text{ if and only if } \mathbf{m}^\star_{d\overline{a}} \le \mathbf{m}_{d(a-1)}.$$

Note that

$$\mathbf{m}^\star_{d\overline{a}} = ha\left(\lfloor \tfrac{\overline{a}}{k-1} \rfloor + 2\right) + d\overline{a} > ha\left(\lfloor \tfrac{a-2}{k-1} \rfloor + 1\right) + d(a-1) = \mathbf{m}_{d(a-1)} \tag{7}$$

implies $\lfloor \frac{\overline{a}}{k-1} \rfloor = \lfloor \frac{a-2}{k-1} \rfloor$. Since we have both $(k-1) \mid (\overline{a}+1)$ and $\overline{a} \le a-1$, we must have $(k-1) \mid a$ or $(k-1) \mid (a-1)$. If $(k-1) \mid a$, then $\overline{a} = a-1$ and Eq. (7) holds. If $(k-1) \mid (a-1)$, then $\overline{a} = a-2$ and Eq. (7) reduces to $ha > d$. This completes the description of A^- in this case. □

Theorem 3.4 *Let a, d, h be positive integers, where $a \ge 3$ and $\gcd(a,d) = 1$. Let $A = MAP(a,d,h,a)$. Then*

$$A^- = \left\{ha + jd : 2 \le j \le a - 2 - \left\lfloor \tfrac{ha}{d} \right\rfloor\right\}.$$

Proof By Proposition 3.2, $ha + jd \notin A^-$ for $j \in \{0, 1, a-1\}$. Fix $j \in \{2, \ldots, a-2\}$, and let $A^\star = A \setminus \{ha + j\}$. For $x \in \{1, \ldots, a-1\}$, let $\mathbf{m}^\star_{dx}$ denote the least positive integer in $\langle A^\star \rangle$ congruent to dx modulo a. Since $\mathbf{m}_{dx} = ha + dx$ by Lemma 2.1, $\mathbf{m}^\star_{dx} \ge \mathbf{m}_{dx}$ and $ha + dx \in A^\star$ for $x \ne j$, we have $\mathbf{m}^\star_{dx} = \mathbf{m}_{dx}$ whenever $x \ne j$.

On the other hand, $(ha+d) + (ha + (j-1)d) = 2ha + jd \in \langle A^\star \rangle$ and $ha + jd \notin \langle A^\star \rangle$. Hence $\mathbf{m}^\star_{dj} = 2ha + jd$, so that $\mathrm{F}(\langle A^\star \rangle) = \max\{d(a-1) + (h-1)a, (2h-1)a + jd\}$. Since $\mathrm{F}(\langle A \rangle) = d(a-1) + (h-1)a$ by Eq. (1), $ha + jd \in A^-$ if and only if $ha + jd \le d(a-1)$. This is not possible when $d = 1$; for $d > 1$, this is equivalent to $j \le a - 1 - \frac{ha}{d}$. Since $d \nmid a$ except when $d = 1$, $j \le a - 2 - \left\lfloor \frac{ha}{d} \right\rfloor$. □

We summarize the results in this section that describe the set A^- in all cases for $A = \mathrm{MAP}(a,d,h,k)$, where a, d, h, k are positive integers, $\gcd(a,d) = 1$, $h \ge 1$ and $3 \le k \le a$.

Theorem 3.5 *Let a, d, h, k be positive integers, where $\gcd(a,d) = 1$ and $3 \le k \le a$. Let $A = MAP(a,d,h,k)$. Then*

(i) *If $k = 3$, then $A^- = \emptyset$.*
(ii) *If $4 \le k \le a-1$, then*

$$A^- = \{ha + jd : 2 \le j \le k-3\} \bigcup S \bigcup T,$$

where

$$S = \begin{cases} \emptyset & \text{if } (k-1) \mid a \text{ or } (k-1) \mid (a-1), d < a; \\ \{ha + (k-2)d\} & \text{otherwise}, \end{cases}$$

and

$$T = \begin{cases} \{ha + (k-1)d\} & \text{if } \lfloor \frac{a-2}{k-1} \rfloor = \lfloor \frac{a-2}{k-2} \rfloor; \\ \emptyset & \text{if } \lfloor \frac{a-2}{k-1} \rfloor \neq \lfloor \frac{a-2}{k-2} \rfloor. \end{cases}$$

(iii) *If* $k = a \geq 5$, *then*

$$A^- = \left\{ha + jd : 2 \leq j \leq a - 2 - \left\lfloor \tfrac{ha}{d} \right\rfloor\right\}.$$

(iv) $ha + (k-1)d \in A^-$ *if and only if* $\left\lfloor \frac{a-2}{k-1} \right\rfloor = \left\lfloor \frac{a-2}{k-2} \right\rfloor$.

Parts (i) and (iv) are from Proposition 3.2, part (ii) from Proposition 3.3, and part (iii) from Proposition 3.4.

4 Maximum deletion

Let $A = \text{MAP}(a, d, h, k)$, $\gcd(a, d) = 1$, $h \geq 1$ and $3 \leq k \leq a$. From Sect. 3, we have a characterization of single element removals from A that leave the Frobenius number unchanged. It is a natural question to ask for the size of the largest subset of A that can be removed from A without changing the Frobenius number, and then to characterize such sets. This amounts to asking for the subset of least size which has the Frobenius number $\text{F}(\langle A \rangle)$, and then to characterize those subsets.

For a fixed values of a, d, h, k with restrictions as above, Ritter [7] denotes the set $A = \text{MAP}(a, d, h, k)$ by A_k, and by ℓ_k the size of the largest subset of A_k that may be removed from A_k without changing the Frobenius number. He shows that

$$1 - \frac{4}{\sqrt{k}} \leq \frac{\ell_k}{k} \leq 1 - \frac{3}{k} \tag{8}$$

provided $k < a$, or $k = a$ with $d > 2h\sqrt{k}$. Ritter uses the result

$$\begin{aligned} &\text{F}\,(a, a+d, a+2d, \ldots, a+(k-1)d) \\ &\quad = \text{F}\,(a, a+d, a+(k-1)d) \text{ provided } k = a \text{ and } d > k(k-3) \end{aligned}$$

of Selmer [11], together with an algorithm by Rødseth [10], to prove the upper bound in Eq. (8). The lower bound in Eq. (8) can be improved to $1 - 4/k$ if $a > k(k-4) + 3$. Moreover, he determines some extremal sets for which these bounds are achieved.

Let $A = \text{MAP}(a, d, h, k)$. If A' is a subset of A with the same Frobenius number as A, then Mendelsohn [5], and later Selmer [11], showed that $|A'| \geq 3$. This also follows from Proposition 3.2, since both a and $a + hd$ must belong to A' and $\text{F}(\langle a, ha + d \rangle) >$

$\mathrm{F}(\langle A\rangle)$, as will be shown in the proof of Theorem 4.2. We begin this Section by characterizing 3-sets A' which have the same Frobenius number as A in Theorem 4.2.

Definition 4.1 Let $\mathscr{A}$ denote the collection of all sets $A = \mathrm{MAP}(a, d, h, k)$, where a, d, h are fixed, $\gcd(a, d) = 1$, $h \geq 1$ and k runs over the set $\{3, \ldots, a\}$. Then $\overline{A^-}$ is any subset of $A \in \mathscr{A}$ satisfying the following two conditions:

(i) $\mathrm{F}\left(\langle \overline{A^-}\rangle\right) = \mathrm{F}(\langle A\rangle)$;
(ii) if A' is any subset of A such that $\mathrm{F}(\langle A'\rangle) = \mathrm{F}(\langle A\rangle)$, then $|A'| \geq |\overline{A^-}|$.

Theorem 4.2 *Let $A = MAP(a, d, h, k)$, where $\gcd(a, d) = 1$, $h \geq 1$ and $3 \leq k \leq a$. Then*

(i) $|\overline{A^-}| \geq 3$.
(ii) *Suppose $|\overline{A^-}| = 3$. Then*

$$\overline{A^-} = \{a, ha + d, ha + jd\}, 2 \leq j \leq k - 1,$$

and exists if and only if $\lfloor \frac{a-2}{j} \rfloor = \lfloor \frac{a-2}{k-1} \rfloor$ and either

$$j \mid (a-1), d > ha(j-2) \quad or \quad j \mid (a-2), d > \tfrac{1}{2}ha(j-3)$$

holds.

Proof We know that $a, ha + d \in \overline{A^-}$ by Proposition 3.2. Moreover, since

$$\begin{aligned}\mathrm{F}(\langle a, ha + d\rangle) &= a(ha + d) - ((h + 1)a + d) = ha(a - 1) + d(a - 1) - a \\ &> ha\left(\lfloor \tfrac{a-2}{k-1} \rfloor + 1\right) + d(a - 1) - a = \mathrm{F}(\langle A\rangle),\end{aligned}$$

we must have $|\overline{A^-}| > 2$. Thus, there must exist $ha + jd \in \overline{A^-}$ for some $j \in \{2, \ldots, k - 1\}$.

Suppose $A^\star = \{a, ha + d, ha + jd\}$ satisfies $\mathrm{F}(\langle A^\star\rangle) = \mathrm{F}(\langle A\rangle)$ for some $j \in \{2, \ldots, k - 1\}$. Since

$$\mathrm{F}(\langle A^\star\rangle) \geq \mathrm{F}(\langle a, ha + d, ha + 2d, \ldots, ha + jd\rangle) \geq \mathrm{F}(\langle A\rangle),$$

we must have $\mathrm{F}(\langle a, ha + d, ha + 2d, \ldots, ha + jd\rangle) = \mathrm{F}(\langle A\rangle)$. This is equivalent to $\lfloor \frac{a-2}{j} \rfloor = \lfloor \frac{a-2}{k-1} \rfloor$, by Eq. (1).

Since $\mathrm{F}(\langle a, ha + d, ha + jd\rangle)$ is of the form $(ha + d)x + (ha + jd)y - a$ with $x \geq 0$ and $y \geq 0$, we must have $x_1 \geq 0$ and $x_j \geq 0$ satisfying

$$ha\left(\lfloor \tfrac{a-2}{j} \rfloor + 1\right) + d(a-1) = (ha+d)x_1 + (ha+jd)x_j = ha(x_1 + x_j) + d(x_1 + jx_j). \tag{9}$$

Reducing Eq. (9) modulo a yields $x_1 + jx_j \equiv -1 \pmod{a}$ since $\gcd(a, d) = 1$. Thus, $x_1 + jx_j = a - 1 + at$, with $t \geq 0$. Using this in Eq. (9) gives $x_1 + x_j = \lfloor \frac{a-2}{j} \rfloor + 1 - dt$. But now

$$\begin{aligned}(j-1)x_1 &= j(x_1 + x_j) - (x_1 + jx_j) \\ &= j\left(\lfloor \tfrac{a-2}{j} \rfloor + 1 - dt\right) - (a - 1 + at) \leq (j-1) - (a + jd)t.\end{aligned}$$

This implies $t \leq 0$ since $x_1 \geq 0$. Hence, $t = 0$, and we have $x_1 \leq 1$ and $x_1 + jx_j = a - 1$. Therefore, if $x_1 = 0$, then $j \mid (a-1)$, and if $x_1 = 1$, then $j \mid (a-2)$.

For $x \in \{1, \ldots, a-1\}$, let $\mathbf{m}^{\star}_{dx}$ denote the least positive integer in $\langle A^{\star} \rangle$ congruent to dx modulo a. Let $\overline{j}$ be one less that the largest multiple of j that does not exceed $a - 1$, i.e., let $\overline{j} = j\lfloor \frac{a-1}{j} \rfloor - 1$. Thus, $\mathbf{m}^{\star}_{d\overline{j}}$ is the least positive integer of the form $ma + d\overline{j}$ for which there exist $x_1 \geq 0$ and $x_j \geq 0$ satisfying

$$ma + d\overline{j} = (ha + d)x_1 + (ha + jd)x_j = ha(x_1 + x_j) + d(x_1 + jx_j). \tag{10}$$

Reducing modulo a yields $x_1 + jd_j \equiv \overline{j} \pmod{a}$ since $\gcd(a, d) = 1$. Hence $x_1 + jx_j = \overline{j} + at$ for some $t \geq 0$, and so $m = h(x_1 + x_j) + dt$. In order to minimize m, we must choose $x_j = \lfloor \frac{\overline{j}+at}{j} \rfloor$ and $x_1 = (\overline{j} + at) \pmod{j}$. So if we write

$$f(t) = h\left(\lfloor \tfrac{at-1}{j} \rfloor + (at - 1) \mod j\right) + dt, t \geq 0, \tag{11}$$

then $f(t + j) = f(t) + ha + dj$ for each $t \geq 0$. Therefore, the minimum value of m is given by

$$\begin{aligned}m_0 &= \min_{0 \leq t \leq j-1} \left(h\left(\lfloor \tfrac{\overline{j}+at}{j} \rfloor + (\overline{j} + at) \mod j\right) + dt\right) \\ &= h\lfloor \tfrac{a-1}{j} \rfloor + \min_{0 \leq t \leq j-1} \left(h\left(\lfloor \tfrac{at-1}{j} \rfloor + (at - 1) \mod j\right) + dt\right).\end{aligned} \tag{12}$$

Case (i). ($j \mid (a-1)$) Since $at - 1 \equiv t - 1 \pmod{j}$, we have

$$f(t) = h\left(\tfrac{a-1}{j}t + \lfloor \tfrac{t-1}{j} \rfloor + (t - 1) \mod j\right) + dt$$

from Eq. (11). Thus $f(t + 1) > f(t)$ for $t > 0$, so that $\min_{0 \leq t \leq j-1} f(t) = \min\{f(0), f(1)\}$. If $h\frac{a-1}{j} + d = f(1) < f(0) = h(j - 2)$, we must have $m_0 a + d\overline{j} = \mathbf{m}^{\star}_{d\overline{j}} \leq \mathbf{m}_{d(a-1)}$, with $m_0 = h\frac{a-1}{j} + f(1)$ and $\overline{j} = a - 2$. Thus,

$$a\left(2h\tfrac{a-1}{j} + d\right) + d(a-2) \leq ha\left(\lfloor \tfrac{a-2}{j} \rfloor + 1\right) + d(a-1) = ha\tfrac{a-1}{j} + d(a-1).$$

The falsity of the above inequality implies $f(0) < f(1)$, so that $m_0 a + d\overline{j} = \mathbf{m}^{\star}_{d\overline{j}} \leq \mathbf{m}_{d(a-1)}$, with $m_0 = h\frac{a-1}{j} + f(0)$. Thus,

$$ha\left(\tfrac{a-1}{j} + j - 2\right) + d(a-2) \leq ha\tfrac{a-1}{j} + d(a-1),$$

which is equivalent to $ha(j-2) \leq d$.
Case (ii). ($j \mid (a-2)$) Since $at - 1 \equiv 2t - 1 \pmod{j}$, we have

$$f(t) = h\left(\tfrac{a-2}{j}t + \lfloor\tfrac{2t-1}{j}\rfloor + (2t-1) \mod j\right) + dt$$

from Eq. (11). Thus $f(t+1) > f(t)$ for $0 < t < \frac{j-1}{2}$ and for $\frac{j}{2} < t \leq j-1$, and so $\min_{0\leq t\leq j-1} f(t) = \min\{f(0), f(1), f\left(\lceil\frac{j+1}{2}\rceil\right)\}$. Moreover,

$$f\left(\lceil\tfrac{j+1}{2}\rceil\right) \geq h\tfrac{a}{2} + d\left\lceil\tfrac{j+1}{2}\right\rceil > h\left(\tfrac{a-2}{j} + 1\right) + d = f(1).$$

If $h\left(\frac{a-2}{j} + 1\right) + d = f(1) < f(0) = h(j-2)$, we must have $m_0 a + d\overline{j} = \mathbf{m}^{\star}_{d\overline{j}} \leq \mathbf{m}_{d(a-1)}$, with $m_0 = h\frac{a-2}{j} + f(1)$ and $\overline{j} = a - 3$. Thus,

$$a\left(2h\tfrac{a-2}{j} + d + h\right) + d(a-3) \leq ha\left(\lfloor\tfrac{a-2}{j}\rfloor + 1\right) + d(a-1) = ha\left(\tfrac{a-2}{j} + 1\right) + d(a-1).$$

The falsity of the above inequality implies $f(0) < f(1)$, so that $m_0 a + d\overline{j} = \mathbf{m}^{\star}_{d\overline{j}} \leq \mathbf{m}_{d(a-1)}$, with $m_0 = h\frac{a-2}{j} + f(0)$. Thus,

$$ha\left(\tfrac{a-2}{j} + j - 2\right) + d(a-3) \leq ha\left(\tfrac{a-2}{j} + 1\right) + d(a-1),$$

which is equivalent to $ha(j-3) \leq 2d$.
The equality in both cases may be neglected since $\gcd(ha, d) = 1$.

Conversely, we show that $\mathrm{F}(\langle A^{\star}\rangle) = \mathrm{F}(\langle A\rangle)$, where $A^{\star} = \{a, ha + d, ha + jd\}$ for j meeting the conditions in the statement of this theorem. Fix $x \in \{1, \ldots, a-1\}$. Since

$$(ha + jd)\lfloor\tfrac{x}{j}\rfloor + (x \mod j)(ha + d) = ha\left(\lfloor\tfrac{x}{j}\rfloor + x \mod j\right) + dx \in \langle A^{\star}\rangle,$$

we have $\mathbf{m}^{\star}_{dx} \leq ha\left(\lfloor\frac{x}{j}\rfloor + x \bmod \mathrm{j}\right) + dx$. Therefore

$$\max_{1\leq x\leq a-1} \mathbf{m}^{\star}_{dx} \leq \max\left\{ha\left(\lfloor\tfrac{\overline{j}}{j}\rfloor + j - 1\right) + d\overline{j}, ha\left(\lfloor\tfrac{a-1}{j}\rfloor + a - 1 \mod j\right) + d(a-1)\right\}, \tag{13}$$

where $\overline{j} = j\lfloor\frac{a-1}{j}\rfloor - 1$, as defined before.

If $j \mid (a-1)$, then $\overline{j} = a-2$, and the two terms in the set in Eq. (13) are $ha\left(\frac{a-1}{j}+j-2\right)+d(a-2)$ and $ha\frac{a-1}{j}+d(a-1)$. If $d > ha(j-2)$, then the second expression is larger, and equals $\mathbf{m}_{d(a-1)}$ in this case.

If $j \mid (a-2)$, then $\overline{j} = a-3$, and the two terms in the set in Eq. (13) are $ha\left(\frac{a-2}{j}+j-2\right)+d(a-3)$ and $ha\left(\frac{a-2}{j}+1\right)+d(a-1)$. If $2d > ha(j-3)$, then the second expression is larger, and equals $\mathbf{m}_{d(a-1)}$ in this case.

This completes the proof of the theorem. □

Remark 4.3 For $A = \mathrm{MAP}(a,d,1,a)$ and $A' = \{a, a+d, a+(a-1)d\}$, Selmer [11] has shown that $\mathrm{F}(\langle A'\rangle) = \mathrm{F}(\langle A\rangle)$ provided $d > a(a-3)$. From Theorem 4.2, for $A = \mathrm{MAP}(a,d,h,a)$ and $A' = \{a, ha+d, ha+(a-1)d\}$ we have $\mathrm{F}(\langle A'\rangle) = \mathrm{F}(\langle A\rangle)$ exactly when $d > ha(a-3)$. Therefore, the bound of Selmer cannot be improved.

Theorem 4.2 shows the existence of a 3-set A' with the same Frobenius number as A. We next consider 4-sets A' with the same property. We know that any such set must contain a and $ha+d$. Moreover, by an argument in the beginning of the proof of Theorem 4.2, we may assume that $ha+(k-1)d$ also belongs to A'. Assuming the additional element in A' is $ha+(k-2)d$, we show that A' has the same Frobenius number as A precisely when $a \geq (k-2)(k-3)$ in Theorem 4.4.

Theorem 4.4 *Let $A = MAP(a,d,h,k)$ and $A' = \{a, ha+d, ha+(k-2)d, ha+(k-1)d\}$, where $\gcd(a,d) = 1$, $h \geq 1$ and $5 \leq k \leq a$.*

(i) *If $a \geq (k-2)(k-3)$, then $F(\langle A'\rangle) = F(\langle A\rangle)$.*
(ii) *If $a = (k-2)(k-3)-1$, then*

$$F(\langle A'\rangle) = \max\left\{ha(k-3)+d(a-k+2), ha(k-4)+d(a-1)\right\} - a$$
$$= \begin{cases} ha(k-3)+d(a-k+2)-a & \text{if } ha > d(k-3); \\ ha(k-4)+d(a-1)-a & \text{if } ha < d(k-3). \end{cases}$$

In particular, $F(\langle A'\rangle) > F(\langle A\rangle)$ if and only if $ha > d(k-3)$.

Proof For $x \in \{1,\ldots,a-1\}$, let $\mathbf{m}_x$, $\mathbf{m}'_x$ denote the least positive integer congruent to x modulo a in $\langle A\rangle$, $\langle A'\rangle$, respectively. Recall from Lemma 2.1 and Eq. (1) that

$$\mathbf{m}_{dx} = ha\left\lceil\tfrac{x}{k-1}\right\rceil + dx \quad \text{and} \quad \mathrm{F}(\langle A\rangle) = ha\left\lceil\tfrac{a-1}{k-1}\right\rceil + d(a-1) - a = \mathbf{m}_{d(a-1)} - a.$$

Since $\mathrm{F}(\langle A'\rangle) \geq \mathrm{F}(\langle A\rangle)$ and $\gcd(a,d) = 1$, we have $\max_{1\leq x\leq a}\mathbf{m}'_{dx} \geq \mathbf{m}_{d(a-1)}$.
We claim that $\mathbf{m}'_{dx} \leq \mathbf{m}_{d(a-1)}$ for $x \in \langle k-2, k-1\rangle$. To determine $\mathbf{m}'_{dx}$, we need to find the least nonnegative integer m for which there exist $x_1 \geq 0$, $x_{k-2} \geq 0$ and $x_{k-1} \geq 0$ satisfying

$$\begin{aligned} ma + dx &= (ha+d)x_1 + (ha+(k-2)d)\,x_{k-2} + (ha+(k-1)d)\,x_{k-1} \\ &= ha(x_1+x_{k-2}+x_{k-1}) + d\,(x_1+(k-2)x_{k-2}+(k-1)x_{k-1}). \end{aligned} \quad (14)$$

If $x \in \langle k-2, k-1\rangle$, we can choose nonnegative integers r, s, with $0 \le s < k-1$, such that $x = r(k-1) + s(k-2)$. Since $x = (r+s)(k-1) - s \le a-1$, we have $r+s \le \left\lceil \frac{a-1}{k-1} \right\rceil$. With $x_1 = 0, x_{k-2} = s, x_{k-1} = r$, we have $m = h(r+s) \le h\left\lceil \frac{a-1}{k-1} \right\rceil$ and so $\mathbf{m}'_{dx} \le \mathbf{m}_{d(a-1)}$. This proves the claim.
Therefore, $\max_{1\le x\le a} \mathbf{m}'_{dx}$ is achieved at some value of $x \notin \langle k-2, k-1\rangle$.

(i) Suppose $a \ge (k-2)(k-3) = (k-1)(k-4) + 2$. If $x \notin \langle k-2, k-1\rangle$, then $x = \mathrm{F}(\langle k-2, k-1\rangle) - x'$ for some $x' \in \langle k-2, k-1\rangle$. Since $\mathrm{F}(\langle k-2, k-1\rangle) = (k-3)(k-2) - 1 = (k-1)(k-4) + 1$, writing $x' = r(k-1) + s(k-2) = (r+s)(k-1) - s$, where r, s are nonnegative integers and $0 \le s < k-1$, we have $x = (k-4-r-s)(k-1) + s + 1$. Note that $k-4-r-s \ge 0$ since $s+1 \le k-1$ and $x > 0$. The choice $x_{k-1} = k-4-r-s, x_1 = s+1, x_{k-2} = 0$ yields the sum $x_1 + x_{k-2} + x_{k-1} = k-3-r$ (the sum is lesser if $s+1 = k-2$ or $k-1$), and so $m \le h(k-3-r) \le h(k-3) \le h\left\lceil \frac{a-1}{k-1} \right\rceil$. Thus, $\mathbf{m}'_{dx} \le \mathbf{m}_{d(a-1)}$, so that part (i) is proved.

(ii) Suppose $a = (k-2)(k-3) - 1 = (k-1)(k-4) + 1$, so that $\mathbf{m}_{d(a-1)} = ha(k-4) + d(a-1)$. If $x \notin \langle k-2, k-1\rangle$, then $x = a - x'$ for some $x' \in \langle k-2, k-1\rangle$ since $a = \mathrm{F}(\langle k-2, k-1\rangle)$. Arguing as in part (i), there exists nonnegative integers r, s, $0 \le s < k-1$ such that we choose $x_{k-1} = k-4-r-s$, $x_1 = s+1$, $x_{k-2} = 0$ in the expression in Eq. (14). If this choice minimizes $x_1 + x_{k-2} + x_{k-1}$, $m = k-3-r$ and so $\mathbf{m}'_{dx} = ha(k-3-r) + dx$.
If $r > 0$, then $\mathbf{m}'_{dx} < \mathbf{m}_{d(a-1)}$. Thus, $\max_{1\le x\le a} \mathbf{m}'_{dx}$ is achieved at $x = a - s(k-2)$, $s \ge 1$. Since $\mathbf{m}'_{dx} = ha(k-3) + dx$ is independent of s for all such x,

$$\max_{1\le x\le a-1} \mathbf{m}'_{dx} = \mathbf{m}'_{d(a-k+2)} = ha(k-3) + d(a-k+2).$$

Since $\max_{1\le x\le a-1} \mathbf{m}'_{dx} \ge \mathbf{m}_{d(a-1)}$, part (ii) is proved.

□

Remark 4.5 Ritter [7, Theorem 2] shows that $\mathrm{F}(\langle A'\rangle) = \mathrm{F}(\langle A\rangle)$ for A, A' in Theorem 4.4 provided $a > (k-1)(k-3)$. From Theorem 4.4, we see that $\mathrm{F}(\langle A'\rangle) = \mathrm{F}(\langle A\rangle)$ provided $a \ge (k-2)(k-3)$. Furthermore, we show that $\mathrm{F}(\langle A'\rangle) > \mathrm{F}(\langle A\rangle)$ provided $a = (k-2)(k-3) - 1$ and $ha > d(k-3)$. We note that $ha = d(k-3)$ is not possible since $\gcd(a, d) = 1$ and $1 < k-3 < a$.

Prompted by the lower bound in Eq. (8), we explore the possibility of finding sets A' with the same Frobenius number as A, and of cardinality roughly $\sqrt{k}$ for some particular choices of k. Lagrange's four square theorem is the statement that every positive integer is a sum of at most four positive squares. By $\mathbf{s}(n)$ we mean the least positive integer t such that n is a sum of t positive squares with elements from $\{1, \ldots, k-1\}$.

Theorem 4.6 *Let $A = MAP(a, d, h, a)$ and $A' = \{a, ha + (a-1)d\} \cup \{ha + j^2 d : 1 \le j^2 \le a-1\}$, where $\gcd(a, d) = 1$ and $h \ge 1$. Then*

$$F(\langle A'\rangle) = F(\langle A\rangle) \tag{15}$$

if $d > 3ha$.
Moreover, if d_0 denotes the least positive integer such that Eq. (15) holds for all $d > d_0$, then

$$d_0 = \left(\max_{1 \le x \le a-2} \frac{\boldsymbol{s}(x) - 1}{a - 1 - x} \right) ha$$

$$= \begin{cases} 3ha & \text{if and only if } \boldsymbol{s}(a-2) = 4; \\ 2ha & \text{if and only if } \boldsymbol{s}(a-2) = 3; \\ \frac{3}{2}ha & \text{if and only if } \boldsymbol{s}(a-2) \le 2 \text{ and } \boldsymbol{s}(a-3) = 4; \\ ha & \text{if and only if } (\boldsymbol{s}(a-2) = 2, \boldsymbol{s}(a-3) \le 3) \text{ or } (\boldsymbol{s}(a-2) \le 2, \boldsymbol{s}(a-3) = 3) \\ & \text{or } (\boldsymbol{s}(a-2) \le 2, \boldsymbol{s}(a-3) \le 3, \boldsymbol{s}(a-4) = 4)\,. \end{cases}$$

Proof (i) For $x \in \{1, \ldots, a-1\}$, let $\mathbf{m}_x, \mathbf{m}'_x$ denote the least positive integer congruent to x modulo a in $\langle A \rangle$, $\langle A' \rangle$, respectively. Recall from Lemma 2.1 and Eq. (1) that

$$\mathbf{m}_{dx} = ha + dx \quad \text{and} \quad \mathrm{F}(\langle A \rangle) = \mathbf{m}_{d(a-1)} - a.$$

Since $\mathrm{F}(\langle A' \rangle) \ge \mathrm{F}(\langle A \rangle)$, equality would follow from $\mathbf{m}'_{dx} \le \mathbf{m}_{d(a-1)}$ for each $x \in \{1, \ldots, a-1\}$.
Note that $\mathbf{m}'_{d(a-1)} = ha + d(a-1) = \mathbf{m}_{d(a-1)}$. Since every positive integer is a sum of at most four positive squares, with $x = x_1^2 + x_2^2 + x_3^2 + x_4^2$, we have

$$\begin{aligned} \mathbf{m}'_{dx} &\le (ha + dx_1^2) + \cdots + (ha + dx_4^2) = 4ha + dx \le 3ha + (ha + d(a-2)) \\ &< ha + d(a-1) = \mathbf{m}_{d(a-1)} \end{aligned}$$

for $1 \le x \le a-2$ if $d > 3ha$.
Thus, $\mathbf{m}'_{dx} \le \mathbf{m}_{d(a-1)}$ for each $x \in \{1, \ldots, a-1\}$, so that $\mathrm{F}(\langle A' \rangle) \le \mathrm{F}(\langle A \rangle)$. Together with the obvious reverse inequality, we get $\mathrm{F}(\langle A' \rangle) = \mathrm{F}(\langle A \rangle)$ if $d > 3ha$.
(ii) Note that $\mathbf{m}'_{dx} = ha + dx$ for $x = j^2$, $1 \le j^2 \le a-1$, and for $x = a-1$. For all other values of $x \in \{1, \ldots, a-1\}$, we require at least $\mathbf{s}(x)$ squares from the collection $\{j^2 : 1 \le j^2 \le a-1\}$ to sum to obtain x. Therefore, $\mathbf{m}'_{dx} = \mathbf{s}(x) \cdot ha + dx$, and so Eq. (15) holds if and only if

$$\mathbf{s}(x) \cdot ha + dx \le ha + d(a-1) \tag{16}$$

for $1 \le x \le a-2$. Rearranging Eq. (16), we get

$$d_0 = \left(\max_{1 \le x \le a-2} \frac{\mathbf{s}(x) - 1}{a - 1 - x} \right) ha. \tag{17}$$

Let $M = \max_{1 \le x \le a-2} (\mathbf{s}(x) - 1)/(a - 1 - x)$. Then $M \le 3$, with equality if and only if $\mathbf{s}(a-2) = 4$ with the maximum achieved at $x = a-2$.
Now assume $\mathbf{s}(a-2) \le 3$. Then $M \le 2$, with equality if and only if $\mathbf{s}(a-2) = 3$.
Next assume $\mathbf{s}(a-2) \le 2$. Then $M \le 3/2$, with equality if and only if $\mathbf{s}(a-3) = 4$ with maximum achieved at $x = a-3$.

If $M < 3/2$, then $M \leq 1$, and equality occurs if and only if either (A) $\mathbf{s}(a-2) = 2$, $\mathbf{s}(a-3) \leq 3$ or (B) $\mathbf{s}(a-2) \leq 2$, $\mathbf{s}(a-3) = 3$ or (C) $\mathbf{s}(a-2) \leq 2$, $\mathbf{s}(a-3) \leq 3$, $\mathbf{s}(a-4) = 4$ hold.
The first three cases are pairwise disjoint, and each of the three subcases (A), (B), (C) of the fourth case is disjoint with the first three cases. On the other hand, subcases (A), (B), (C) overlap but yield the same result. Finally, the only tuples $(\mathbf{s}(a-2), \mathbf{s}(a-3), \mathbf{s}(a-4))$ not covered by any of the four cases are:

$$(x, y, z), \quad (1, 2, 2), \quad (1, 2, 3),$$

where at least two of x, y, z equal 1. The first possibility is ruled out because two consecutive squares must differ by at least 3. To rule out the second possibility, note that $a-2$ is either 0 or 1 mod 4, and then $a-3$ or $a-4$ is 3 mod 4, and so cannot be a sum of two squares. To rule out the third possibility, $a-2$ must be 1 mod 8 (since $a-3$ is a sum of two squares); but then $a-4$ is 7 mod 8, and so cannot be a sum of three squares.
This completes the proof.

□

Theorem 4.7 *Let* $A = MAP(a, d, h, a-1)$ *and* $A' = \{a, ha+(a-2)d\} \cup \{ha + j^2 d : 1 \leq j^2 \leq a-2\}$, *where* $\gcd(a, d) = 1$ *and* $h \geq 1$. *Then*

$$F(\langle A' \rangle) = F(\langle A \rangle)$$

if $d > ha$.
Moreover, if d_0 *denotes the least positive integer such that Eq.* (15) *holds for all* $d > d_0$, *then*

$$d_0 = \left(\max_{1 \leq x \leq a-2} \frac{\mathbf{s}(x) - 2}{a - 1 - x} \right) ha$$

$$= \begin{cases} ha & \text{if and only if } \mathbf{s}(a-3) = 4; \\ \frac{2}{3}ha & \text{if and only if } \mathbf{s}(a-3) \leq 3 \text{ and } \mathbf{s}(a-4) = 4; \\ \frac{1}{2}ha & \text{if and only if } \mathbf{s}(a-3) = 3, \mathbf{s}(a-4) \leq 3 \text{ or } \mathbf{s}(a-3) \leq 3, \mathbf{s}(a-4) \\ & \quad \leq 3, \mathbf{s}(a-5) = 4; \\ \frac{2}{5}ha & \text{if and only if } \mathbf{s}(a-3) \leq 2, \mathbf{s}(a-4) \leq 3, \mathbf{s}(a-5) \leq 3 \text{ and } \mathbf{s}(a-6) = 4; \\ \frac{1}{3}ha & \text{if and only if } \mathbf{s}(a-3) \leq 2, \mathbf{s}(a-4) = 3, \mathbf{s}(a-5) \leq 3, \mathbf{s}(a-6) \leq 3. \end{cases}$$

Proof The proof runs along the same lines as in Theorem 4.6. We omit the details. □

Theorem 4.8 *Let* $A = MAP(a, d, h, k)$ *and* $A' = \{a, ha + (k-1)d\} \cup \{ha + j^2 d : 1 \leq j^2 \leq k-1\}$, *where* $\gcd(a, d) = 1$, $h \geq 1$. *If* $k = \left\lfloor \frac{a+1}{2} \right\rfloor$ *or of the form* $a - t^2$, $1 \leq t^2 \leq \frac{a-1}{2}$, *then*

$$F(\langle A' \rangle) = F(\langle A \rangle)$$

for $d > 3ha$.

Proof Let $k = \lfloor \frac{a+1}{2} \rfloor$ or of the form $a - t^2$, $1 \leq t^2 \leq \frac{a-1}{2}$. For $x \in \{1, \ldots, a-1\}$, let $\mathbf{m}_x$, $\mathbf{m}'_x$ denote the least positive integer congruent to x modulo a in $\langle A \rangle$, $\langle A' \rangle$, respectively. Recall from Lemma 2.1 and Eq. (1) that

$$\mathrm{F}(\langle A \rangle) = \mathbf{m}_{d(a-1)} - a = (2h-1)a + d(a-1)$$

except that $\mathrm{F}(\langle A \rangle) = (3h-1)a + d(a-1)$ if a is even and $k = \lfloor \frac{a+1}{2} \rfloor = \frac{a}{2}$. Since $\mathrm{F}(\langle A' \rangle) \geq \mathrm{F}(\langle A \rangle)$, equality would follow from $\mathbf{m}'_{dx} \leq \mathbf{m}_{d(a-1)}$ for each $x \in \{1, \ldots, a-1\}$.

Suppose $1 \leq x < k-1$. Then x can be expressed as a sum of at most four squares, say $x_1^2, \ldots, x_4^2$, from the set $\{j^2 : 1 \leq j^2 < k-1\}$ and each integer of the form $ha + j^2 d \in A'$, we have

$$\begin{aligned}\mathbf{m}'_{dx} &\leq (ha + dx_1^2) + \cdots + (ha + dx_4^2) = 4ha + dx \leq 3ha + (ha + d(a-2)) \\ &< ha + d(a-1) < \mathbf{m}_{d(a-1)}.\end{aligned}$$

Now suppose $k - 1 \leq x < a - 1$. Then $1 \leq x - (k-1) < k - 1$, and so

$$\begin{aligned}\mathbf{m}'_{dx} &\leq (ha + (k-1)d) + \sum_{i=1}^{4} (ha + dx_i^2) = 5ha + dx \leq 4ha + (ha + d(a-2)) \\ &< 2ha + d(a-1) \leq \mathbf{m}_{d(a-1)}.\end{aligned}$$

Thus, $\mathbf{m}'_{dx} \leq \mathbf{m}_{d(a-1)}$ for each $x \in \{1, \ldots, a-2\}$. Hence, it remains to be shown that $\mathbf{m}'_{d(a-1)} \leq \mathbf{m}_{d(a-1)}$ for the two cases (i) $k = \lfloor \frac{a+1}{2} \rfloor$, and (ii) $k = a - t^2$, $1 \leq t^2 \leq \frac{a-1}{2}$.

Case (i) ($k = \lfloor \frac{a+1}{2} \rfloor$) If a is odd, then $\mathbf{m}'_{d(a-1)} = 2\,(ha + (k-1)d) = \mathbf{m}_{d(a-1)}$. If a is even, then $\mathbf{m}'_{d(a-1)} = 2\,(ha + (k-1)d) + (ha + d) = \mathbf{m}_{d(a-1)}$. Thus, $\mathbf{m}'_{d(a-1)} = \mathbf{m}_{d(a-1)}$ in either case.

Case (ii) ($k = a - t^2$, $1 \leq t^2 \leq \frac{a-1}{2}$) In this case, we have $\mathbf{m}'_{d(a-1)} = (ha + (k-1)d) + (ha + t^2 d) = \mathbf{m}_{d(a-1)}$.

Thus, $\mathrm{F}(\langle A' \rangle) \leq \mathrm{F}(\langle A \rangle)$. Together with the obvious reverse inequality, we get $\mathrm{F}(\langle A' \rangle) = \mathrm{F}(\langle A \rangle)$ if $d > 3ha$. □

Remark 4.9 The bound for d in Theorem 4.8 is sharp, in the sense that there exist a such that $\mathrm{F}(\langle A' \rangle) > \mathrm{F}(\langle A \rangle)$ for $d = 3ha - 1$ in each of the two cases discussed therein. For instance, $a = 17$ for the case $k = \lfloor \frac{a+1}{2} \rfloor$ and $a = 41$ for the case $k = a - 4^2$.

Theorem 4.10 *Let $A = MAP(a, d, h, k)$ and $A' = \{a, ha + (k-1)d\} \cup \{ha + j^2 d : 1 \leq j^2 \leq k-1\}$, where $\gcd(a, d) = 1$, $h \geq 1$ and $k \geq (a+1)/2$. If $a - 1$ cannot be written as a sum of two elements from the set $\{j^2 : 1 \leq j^2 \leq k-1\} \cup \{k-1\}$, then*

$$F(\langle A' \rangle) > F(\langle A \rangle).$$

Proof Let $k \geq \frac{a+1}{2}$. For $x \in \{1, \ldots, a-1\}$, let $\mathbf{m}_x$, $\mathbf{m}'_x$ denote the least positive integer congruent to x modulo a in $\langle A \rangle$, $\langle A' \rangle$, respectively. Recall from Lemma 2.1 and Eq. (1) that

$$\mathrm{F}(\langle A \rangle) = \mathbf{m}_{d(a-1)} - a = (2h-1)a + d(a-1).$$

Since $a-1$ cannot be written as a sum of two elements from the set $J = \{j^2 : 1 \leq j^2 \leq k-1\} \cup \{k-1\}$, therefore $\mathbf{m}'_{d(a-1)} \geq (ha + j_1 d) + (ha + j_2 d) + (ha + j_3 d) = 3ha + d(a-1) > \mathbf{m}_{d(a-1)}$, where $j_1, j_2, j_3 \in J$. This shows $\mathbf{m}'_{d(a-1)} > \mathbf{m}_{d(a-1)}$, so that $\mathrm{F}(\langle A' \rangle) > \mathrm{F}(\langle A \rangle)$. □

Corollary 4.11 *Let $A = MAP(a, d, h, k)$ and $A' = \{a, ha + (k-1)d\} \cup \{ha + j^2 d : 1 \leq j^2 \leq k-1\}$, where $\gcd(a, d) = 1$ and $h \geq 1$ and $k \in \{a-2, a-3\}$. Then there exist infinitely many values of a for which*

$$F(\langle A' \rangle) > F(\langle A \rangle).$$

Proof Consider any a such that $a-1$ is not a sum of two squares. For any such a, Theorem 4.10 applies since the difference $(a-1) - (k-1)$ is either 2 or 3. □

Author Contributions Essential coding using python was done by first author. Both authors contributed to research, writing proofs, and editing the manuscript.

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Declarations

Competing interests The authors declare no competing interests.

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