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ORIGINAL PAPER

New Proofs for the Disjunctive Rado Number of the Equations $x_1 - x_2 = a$ and $x_1 - x_2 = b$

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Abstract

Let m, a, b be positive integers, with $\gcd(a,b)=1$. The disjunctive Rado number for the pair of equations $y-x=ma$, $y-x=mb$, is the least positive integer $R=\mathscr{R}_d(ma,mb)$, if it exists, such that every 2-coloring χ of the integers in $\{1,\ldots,R\}$ admits a solution to at least one of $\chi(x)=\chi(x+ma)$, $\chi(x)=\chi(x+mb)$. We show that $\mathscr{R}_d(ma,mb)$ exists if and only if ab is even, and that it equals $m(a+b-1)+1$ in this case. We also show that there are exactly 2^m valid 2-colorings of $[1,m(a+b-1)]$ for the equations $y-x=ma$ and $y-x=mb$, and use this to obtain another proof of the formula for $\mathscr{R}_d(ma,mb)$.

Keywords 2-coloring · Monochromatic solution · Valid coloring · Disjunctive Rado number

Mathematics Subject Classification 05C55 · 05D10

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1 Introduction

In 1916, Schur [12] showed that for every positive integer r, there exists a least positive integer $s = s(r)$ such that for every r-coloring of the integers in the interval [1, s], there exists a monochromatic solution to $x + y = z$ in [1, s]. Schur's Theorem was generalized in a series of results in the 1930's by Rado leading to a complete resolution to the following problem: characterize systems of linear homogeneous equations with integral coefficients $\mathscr{S}$ such that for a given positive integer r, there exists a least positive integer $n = \mathscr{R}(\mathscr{S}; r)$ such that every r-coloring of the integers in the interval [1, n] yields a monochromatic solution to the system $\mathscr{S}$. There has been a growing interest in the determination of the Rado numbers $\mathscr{R}(\mathscr{S}; r)$, particularly when $\mathscr{S}$ is a single equation and $r = 2$; for instance, see [1–6, 8].

The problem of disjunctive Rado numbers was introduced by Johnson and Schaal in [7]. The 2-color disjunctive Rado number for the set of equations $\mathscr{E}_1, \ldots, \mathscr{E}_k$ is the least positive integer N such that any 2-coloring of $\{1, \ldots, N\}$ admits a monochromatic solution to at least one of the equations $\mathscr{E}_1, \ldots, \mathscr{E}_k$. Johnson and Schaal gave necessary and sufficient conditions for the existence of the 2-color disjunctive Rado number for the equations $x_1 - x_2 = a$ and $x_1 - x_2 = b$ for all pairs of distinct positive integers a, b, and also determined exact values when it exists. They also determined exact values for the pair of equations $ax_1 = x_2$ and $bx_1 = x_2$ whenever a, b are distinct positive integers. Lane-Harvard and Schaal [10] determined exact values of 2-color disjunctive Rado number for the pair of equations $ax_1 + x_2 = x_3$ and $bx_1 + x_2 = x_3$ for all distinct positive integers a, b. Sabo, Schaal and Tokaz [11] determined exact values of 2-color disjunctive Rado number for $x_1 + x_2 - x_3 = c_1$ and $x_1 + x_2 - x_3 = c_2$ whenever c_1, c_2 are distinct positive integers. Kosek and Schaal [9] determined the exact value of 2-color disjunctive Rado number for the equations $x_1 + \cdots + x_{m-1} = x_m$ and $x_1 + \cdots + x_{n-1} = x_n$ for all pairs of distinct positive integers m, n.

Let a, b be distinct positive integers. We denote by $\mathscr{R}_d(a, b)$ the 2-color disjunctive Rado number for the equations $x_1 - x_2 = a$ and $x_1 - x_2 = b$. Conditions for existence of $\mathscr{R}_d(a, b)$, as also the exact value of $\mathscr{R}_d(a, b)$, were determined in [7].

Throughout this paper, we work with the equations $y - x = ma$, $y - x = mb$ instead of $x_1 - x_2 = a$, $x_1 - x_2 = b$, and assume that a, b, m are positive integers, with $\gcd(a, b) = 1$.

We first record in Proposition 1 that the disjunctive Rado number $\mathscr{R}_d(ma, mb)$ does not exist when ab is odd. When ab is even, we characterize all valid 2-colorings of $[1, m(a + b - 1) - 1]$ for the pair of equations $y - x = ma$ and $y - x = mb$, and also show that there are exactly 2^m such 2-colorings; see Theorem 1. We use this characterization to show that all 2-colorings of $[1, m(a + b - 1) + 1]$ admit a monochromatic solution to at least one of the equations $y - x = ma$, $y - x = mb$, resulting in a new proof of the formula for $\mathscr{R}_d(ma, mb)$; see Theorem 2. Thus, these two theorems together result in a characterization of all valid 2-colorings of $[1, \mathscr{R}_d(ma, mb) - 1]$. We believe this approach adds a new dimension to proofs involving the exact determination of Rado numbers. Furthermore, in Theorem 3 we give another proof of the formula for $\mathscr{R}_d(ma, mb)$ by explicitly providing a valid 2-coloring of the interval $[1, \mathscr{R}_d(ma, mb) - 1]$ and showing that every 2-coloring of $[1, \mathscr{R}_d(ma, mb)]$ admits a monochromatic solution to at least one of the equations $y - x = ma$, $y - x = mb$. Our proof is significantly shorter than the proof in [7], and also more explicit in its description of a valid 2-coloring of the interval $[1, \mathscr{R}_d(ma, mb) - 1]$ for an arbitrary $m \geq 1$, in terms of integer linear representations $t = \lambda_t a - \mu_t b$ for $t \in [1, a]$. We make this more precise at the end of the paper to support our claim.

2 Main Results

Proposition 1 *Let m, a, b be positive integers, with $a \neq b$, ab odd and* $\gcd(a, b) = 1$. *Then the disjunctive Rado number $\mathscr{R}_d(ma, mb)$ for the pair of equations $y - x = ma$, $y - x = mb$ does not exist.*

Proof Without loss of generality, we assume $a < b$ throughout this proof. Suppose ab is odd. Define $\Delta : \mathbb{N} \to \{0, 1\}$ by

$$\Delta(x) = \left\lceil \frac{x}{m} \right\rceil (\text{mod } 2).$$

Then, for each $x \in \mathbb{N}$, both $\lceil \frac{x+ma}{m} \rceil - \lceil \frac{x}{m} \rceil = a$ and $\lceil \frac{x+mb}{m} \rceil - \lceil \frac{x}{m} \rceil = b$ are odd. Hence, $\Delta(x + ma) \neq \Delta(x)$ and $\Delta(x + mb) \neq \Delta(x)$ for each $x \in \mathbb{N}$, so Δ provides a valid 2-coloring on $\mathbb{N}$ for the pair of equations $y - x = ma$, $y - x = mb$. Thus, $\mathscr{R}_d(ma, mb)$ does not exist if ab is odd. □

Note that ab is even if and only if exactly one of a, b is even, since $\gcd(a, b) = 1$, and so if and only if $a + b$ is odd. Henceforth, we assume ab is even, and therefore that $a + b$ is odd. In Theorem 1, we characterize all valid colorings of $[1, m(a + b - 1)]$ for the pair of equations $y - x = ma$ and $y - x = mb$.

Theorem 1 *Let m, a, b be positive integers, with $a \neq b$, ab even and* $\gcd(a, b) = 1$. *There are exactly 2^m valid 2-colorings of $[1, m(a + b - 1)]$ for the pair of equations $y - x = ma$ and $y - x = mb$.*

Proof Let $\chi : [1, m(a + b - 1)] \to \{0, 1\}$ be a valid 2-coloring for the pair of equations $y - x = ma$ and $y - x = mb$. We only need to define χ on $[1, ma]$ since $\chi(x) \neq \chi(x + ma)$ for every valid 2-coloring.

We claim that χ is completely determined by the m-tuple of 0's and 1's

$$\big(\chi(1), \ldots, \chi(m)\big) = \mathrm{B}_m.$$

When $a = 1$, this defines χ on $[1, ma]$. Therefore, we may assume $a > 1$ for the rest of this proof. Let $t \in [1, a]$, and let $t \leftrightarrow (\lambda_t, \mu_t)$, where $t = \lambda_t b - \mu_t a$, $1 \leq \lambda_t \leq a$. If $\overline{\mathrm{B}_m}$ denotes the complement of B_m, obtained from B_m by interchanging 0s and 1s, and $k \in [1, a - 1]$, we claim that

$$\Big(\chi\big(km+1\big),\ldots,\chi\big((k+1)m\big)\Big)=\begin{cases}\mathrm{B}_m & \text{if } (\lambda_{k+1}-\lambda_1)+(\mu_{k+1}-\mu_1)\text{ is even;}\\ \overline{\mathrm{B}_m} & \text{if } (\lambda_{k+1}-\lambda_1)+(\mu_{k+1}-\mu_1)\text{ is odd.}\end{cases} \tag{1}$$

Let $x \in [1, m]$. For $k \in [1, a-1]$, we show that

$$\chi(x+km)-\chi(x) \equiv (\lambda_{k+1}-\lambda_1)+(\mu_{k+1}-\mu_1)(\text{mod } 2). \tag{2}$$

This is equivalent to the statement of (1).

Since χ is a valid 2-coloring of $[1, m(a+b-1)]$, we have $\chi(s) \neq \chi(s+ma)$ whenever $s, s+ma \in [1, m(a+b-1)]$ and $\chi(s) \neq \chi(s+mb)$ whenever $s, s+mb \in [1, m(a+b-1)]$. So the pair of transformations $s \mapsto s \pm ma$ and the pair of transformations $s \mapsto s \pm mb$ each results in a change in color, as long as the elements stay within the domain of χ.

By an $\langle s_0, s_\ell\rangle$ sequence of length ℓ we mean a sequence $s_0, s_1, s_2, \ldots, s_\ell$ such that $|s_{i+1}-s_i| \in \{ma, mb\}$. An $\langle s_0, s_\ell\rangle$ sequence is a path provided each $s_i \in [1, m(a+b-1)]$.

Every integer in $[m+1, ma]$ is of the form $n = x+km$, with $x \in [1, m]$ and $k \in [1, a-1]$. Two cases arise: (i) $\lambda_{k+1} \leq \lambda_1$, and (ii) $\lambda_{k+1} > \lambda_1$.

Case (i). If $\lambda_{k+1} \leq \lambda_1$, then $\mu_{k+1} \leq \mu_1$ by Lemma 1. We claim that the mappings $s \mapsto s+mb$ (for $1 \leq s \leq m(a-1)$) and $s \mapsto s-ma$ (for $ma < s \leq m(a+b-1)$) provide an $\langle x+km, x\rangle$ path of length $(\lambda_1-\lambda_{k+1})+(\mu_1-\mu_{k+1})$. Since $x = (x+km)-(\mu_1-\mu_{k+1})ma+(\lambda_1-\lambda_{k+1})mb$, it suffices to prove that the appropriate mapping can be applied throughout the sequence starting with $x+km$ and ending with x. Neither of the mappings is possible only when $s \leq ma$ and $s+mb > m(a+b-1)$, or when $m(a-1) < s \leq ma$. Since each mapping preserves elements modulo m, we must show that $x+m(a-1)$ does not lie in the $\langle x+km, x\rangle$ sequence of length $(\lambda_1-\lambda_{k+1})+(\mu_1-\mu_{k+1})$ obtained by applying the appropriate mapping defined above.

If $x+m(a-1) = (x+km)+t_1mb-t_2ma$ for some $t_1, t_2 \in \mathbb{Z}_{\geq 0}$, then $a = k+1+t_1b-t_2a$. Therefore,

$$a = \lambda_{k+1}b-\mu_{k+1}a+t_1b-t_2a = (\lambda_{k+1}+t_1)b-(\mu_{k+1}+t_2)a,$$

so that

$$0 = \lambda_{k+1}+t_1-ta, \quad -1 = \mu_{k+1}+t_2-tb$$

for some $t \in \mathbb{N}$. But then, using Lemma 1,

$$\begin{aligned} t_1+t_2 =& (ta-\lambda_{k+1})+(tb-1-\mu_{k+1}) \\ &\geq (a-\lambda_{k+1})+\big((b-1)-\mu_{k+1}\big) \\ &\geq (\lambda_1-\lambda_{k+1})+(\mu_1-\mu_{k+1}). \end{aligned}$$

Since appropriate applications of the two mappings define an $\langle x+km, x\rangle$ sequence of length $(\lambda_1-\lambda_{k+1})+(\mu_1-\mu_{k+1})$, $x+m(a-1)$ is not a part of this sequence.

Case (ii). If $\lambda_{k+1} > \lambda_1$, then $\mu_{k+1} > \mu_1$ by Lemma 1. We claim that the mappings

$s \mapsto s + ma$ (for $1 \le s \le m(b-1)$) and $s \mapsto s - mb$ (for $mb < s \le m(a+b-1)$) provide an $\langle x+km, x\rangle$ path of length $(\lambda_{k+1} - \lambda_1) + (\mu_{k+1} - \mu_1)$. Since $x = (x+km) + (\mu_{k+1} - \mu_1)ma - (\lambda_{k+1} - \lambda_1)mb$, it suffices to prove that the appropriate mapping can be applied throughout the sequence starting with $x+km$ and ending with x. Neither of the mappings is possible only when $s \le mb$ and $s + ma > m(a+b-1)$, or when $m(b-1) < s \le mb$. Since each mapping preserves elements modulo m, we must show that $x + m(b-1)$ does not lie in the $\langle x+km, x\rangle$ sequence of length $(\lambda_{k+1} - \lambda_1) + (\mu_{k+1} - \mu_1)$ obtained by applying the appropriate mapping defined above.

If $x + m(b-1) = (x+km) + t_1 ma - t_2 mb$ for some $t_1, t_2 \in \mathbb{Z}_{\ge 0}$, then $b - 1 = k + t_1 a - t_2 b$, or $a - 1 = (k+1) - 1 + (t_1+1)a - (t_2+1)b$. Therefore,

$$\begin{aligned}\lambda_{a-1}b - \mu_{a-1}a =& (\lambda_{k+1}b - \mu_{k+1}a) - (\lambda_1 b - \mu_1 a) + (t_1+1)a - (t_2+1)b \\ =& (\lambda_{k+1} - \lambda_1 - t_2 - 1)b - (\mu_{k+1} - \mu_1 - t_1 - 1)a.\end{aligned}$$

From Lemma 1, we have $\lambda_1 + \lambda_{a-1} = a$, so that

$$\lambda_{a-1} = \lambda_{k+1} - \lambda_1 - t_2 - 1 + ta, \quad \mu_{a-1} = \mu_{k+1} - \mu_1 - t_1 - 1 + tb$$

for some $t \in \mathbb{N}$. But then, once again from Lemma 1,

$$\begin{aligned}t_1 + t_2 =& \big(\mu_{k+1} - \mu_1 + (tb - 1 - \mu_{a-1})\big) + \big(\lambda_{k+1} - \lambda_1 + (ta - 1 - \lambda_{a-1})\big) \\ \ge& (\mu_{k+1} - \mu_1) + (\lambda_{k+1} - \lambda_1) + (b - 1 - \mu_{a-1}) + (a - 1 - \lambda_{a-1}) \\ \ge& (\mu_{k+1} - \mu_1) + (\lambda_{k+1} - \lambda_1).\end{aligned}$$

Since appropriate applications of the two mappings define an $\langle x+km, x\rangle$ sequence of length $(\mu_{k+1} - \mu_1) + (\lambda_{k+1} - \lambda_1)$, $x + m(b-1)$ is not a part of this sequence. This proves Eqn. (2).

Since each of $\chi(1), \ldots, \chi(m)$ can be either of 0, 1, B_m assumes any binary m-tuple. So there are 2^m choices for B_m, and so there are at most 2^m valid colorings.

It is easy to see that each such choice of B_m leads to a valid 2-coloring for the pair of equations $y - x = ma$ and $y - x = mb$ using Eqn. (1). Therefore, there are exactly 2^m valid 2-colorings of $[1, m(a+b-1)]$ for the pair of equations $y - x = ma$ and $y - x = mb$. □

Theorem 2 uses the characterization of valid colorings on $[1, m(a+b-1)]$ in Theorem 1 to show that none of these colorings can be extended to $[1, m(a+b-1)+1]$, thereby establishing the value of $\mathscr{R}_d(ma, mb)$. Thus, the two theorems together give a characterization of all valid colorings of $[1, \mathscr{R}_d(ma, mb) - 1]$.

Theorem 2 *Let m, a, b be positive integers, with $a \ne b$, ab even and $\gcd(a,b) = 1$. Then the disjunctive Rado number for the pair of equations $y - x = ma$ and $y - x = mb$ is given by*

$$\mathscr{R}_d(ma, mb) = m(a+b-1) + 1.$$

Proof Theorem 1 characterizes all valid 2-colorings of $[1, m(a+b-1)]$ for the pair of equations $y-x=ma$ and $y-x=mb$. To prove $\mathscr{R}_d(ma, mb) = m(a+b-1)+1$, it suffices to show that regardless of how the domain of χ is extended to include $m(a+b-1)+1$, there must be a monochromatic solution to at least one of $y-x=ma$, $y-x=mb$. Clearly, this can only be possible with $y=m(a+b-1)+1$, and so $x=m(b-1)+1$ or $m(a-1)+1$. Therefore, we must show that $\chi\big(m(a-1)+1\big) \neq \chi\big(m(b-1)+1\big)$ for each valid 2-coloring on $[1, m(a+b-1)]$ given in Theorem 1.

We treat the cases $a=1$ and $a>1$ separately. For $a=1$ and any valid 2-coloring χ of $[1, mb]$ for $y-x=m$, we have $\chi(x+m)-\chi(x) \equiv 1 (\text{mod } 2)$ for $x \in [1, m]$. Thus, $\chi\big(m(b-1)+1\big) - \chi(1) \equiv b-1 \equiv 1(\text{mod } 2)$, since b is even.

For the rest of this proof, assume $a>1$, so that $r>0$. Let χ be any valid 2-coloring of $[1, m(a+b-1)]$ for the pair of equations $y-x=ma$ and $y-x=mb$, characterized in Theorem 1. Since $\chi(x+ma)-\chi(x) \equiv 1(\text{mod } 2)$ for $x \in [1, ma]$, we have $\chi(x+kma)-\chi(x) \equiv k(\text{mod } 2)$ for $1 \leq k \leq q$. In particular, $\chi\big(m(b-1)+1\big) - \chi\big(m(r-1)+1\big) \equiv q(\text{mod } 2)$.

From $ab-(b-1)a = a = \lambda_a b - \mu_a a$ and $r = \lambda_r b - \mu_r a = b - qa$ we have $(\lambda_a, \mu_a) = (a, b-1)$ and $(\lambda_r, \mu_r) = (1, q)$. Therefore, by (2) we have

$$
\begin{aligned}
&\chi\big(m(b-1)+1\big) - \chi\big(m(a-1)+1\big) \\
&= \Big(\chi\big(m(b-1)+1\big) - \chi\big(m(r-1)+1\big)\Big) - \Big(\chi\big(m(a-1)+1\big) - \chi(1)\Big) \\
&+ \Big(\chi\big(m(r-1)+1\big) - \chi(1)\Big) \\
&\equiv q - \big((\lambda_a - \lambda_1) + (\mu_a - \mu_1)\big) + \big((\lambda_r - \lambda_1) + (\mu_r - \mu_1)\big)(\text{mod } 2) \\
&\equiv q - (a-1) - (b-q-1)(\text{mod } 2) \\
&\equiv 1(\text{mod } 2).
\end{aligned}
$$

□

We proceed to give a different proof for the value of $\mathscr{R}_d(ma, mb)$ which relies on the representation of any $t \in \mathbb{Z}$ as an integer linear combination $\lambda_t a - \mu_t b$. The structure of valid 2-colorings is closely connected to these representations: if χ is a valid 2-coloring, then $\chi(\lambda a - \mu b) = 1 - \chi((\lambda+1)a - \mu b) = 1 - \chi(\lambda a - (\mu+1)b)$ when these integers lie in the domain of χ. Lemma 1 helps us establish some structure on these representations.

Lemma 1 *Let m, a, b be positive integers, with $\gcd(a, b) = 1$ and $a < b$. For $t \in [1, a]$, define integers λ_t, μ_t by $t = \lambda_t b - \mu_t a$, $1 \leq \lambda_t \leq a$.*

(i) For $s, t \in [1, a]$, $\lambda_s > \lambda_t$ implies $\mu_s > \mu_t$.
(ii) For $t \in [1, a-1]$, $\lambda_t \leq a-1$ and $\mu_t \leq b-1$.
(iii) For $t \in [1, a]$, $\lambda_t + \lambda_{a-t} = a$.

Proof

(i) Note that $\lambda_s > \lambda_t$ and $\mu_s \leq \mu_t$ leads to the contradiction $s - t = (\lambda_s - \lambda_t)b - (\mu_s - \mu_t)a \geq b > a > s - t$.

(ii) Note that $\lambda_t = a$ implies $t = a$ and $\mu_t = b - 1$. So for $t \neq a$, $\lambda_t \neq \lambda_a = a$ and $\mu_t = \frac{\lambda_t b - t}{a} \leq \frac{ab-1}{a} \leq b - \frac{1}{a}$. Therefore, $\mu_t \leq b - 1$.

(iii) From $a = t + (a - t) = (\lambda_t + \lambda_{a-t})b - (\mu_t + \mu_{a-t})a$ we have $\lambda_t + \lambda_{a-t} = ka$ for some $k \in \mathbb{Z}$. Moreover, $0 < \lambda_t + \lambda_{a-t} < 2a$ implies $k = 1$.

□

Theorem 3 *Let* m, a, b *be positive integers, with* $a \neq b$, ab *even, and* $\gcd(a, b) = 1$. *Then the disjunctive Rado number* $\mathscr{R}_d(ma, mb)$ *for the pair of equations* $y - x = ma$, $y - x = mb$ *is given by*

$$\mathscr{R}_d(ma, mb) = m(a + b - 1) + 1.$$

Proof Without loss of generality, we assume $a < b$ throughout this proof.

I. (Sufficiency for Existence and Upper Bound)

Suppose ab is even. We claim that $\mathscr{R}_d(ma, mb)$ exists, and is bounded above by $m(a + b - 1) + 1$ in this case.

Let $\chi : [1, m(a + b - 1) + 1] \to \{0, 1\}$ be any 2-coloring of $[1, m(a + b - 1) + 1]$. Consider two sequences $\langle x_0, x_1, x_2, \ldots, x_a \rangle$, $\langle y_0, y_1, y_2, \ldots, y_a \rangle$, given by

$$x_k = \left\lceil \frac{kb}{a} \right\rceil ma - kmb + 1, \quad y_k = \left\lceil \frac{(k+1)b}{a} \right\rceil ma - kmb + 1, \quad 0 \leq k \leq a.$$

Note that $1 \leq x_k < y_k$, and

$$\frac{y_k - 1}{m} = \left(\left\lceil \frac{(k+1)b}{a} \right\rceil - \frac{(k+1)b}{a} \right) a + b \leq (a - 1) + b.$$

Thus, each x_k and each y_k lies in the domain of χ.

Suppose, by way of contradiction, that $\chi(x) \neq \chi(x + ma)$ whenever $x, x + ma \in [1, m(a + b - 1) + 1]$ and $\chi(x) \neq \chi(x + mb)$ whenever $x, x + mb \in [1, m(a + b - 1) + 1]$. Since $\chi(x + ma) - \chi(x) \equiv 1 \pmod 2$, we have

$$\chi(y_k) - \chi(x_k) \equiv \left\lceil \frac{(k+1)b}{a} \right\rceil - \left\lceil \frac{kb}{a} \right\rceil \pmod 2$$

for $k \in \{0, \ldots, a\}$. We also have

$$\chi(x_{k+1}) - \chi(y_k) \equiv 1 \pmod 2$$

for $k \in \{0, \ldots, a - 1\}$. But now

$$\begin{aligned}\chi(x_a)-\chi(x_0) &\equiv \sum_{k=0}^{a-1}(\chi(x_{k+1})-\chi(x_k)) \pmod 2\\ &\equiv \sum_{k=0}^{a-1}((\chi(x_{k+1})-\chi(y_k))+(\chi(y_k)-\chi(x_k))) \pmod 2\\ &\equiv \sum_{k=0}^{a-1}\left(1+\left\lceil\frac{(k+1)b}{a}\right\rceil-\left\lceil\frac{kb}{a}\right\rceil\right) \pmod 2\\ &\equiv a+b \pmod 2\\ &\equiv 1 \pmod 2\end{aligned}$$

This contradicts $x_0 = x_a$, thereby proving that every 2-coloring of $[1, m(a+b-1)+1]$ admits a monochromatic solution of either $y-x=ma$ or $y-x=mb$. Thus, $\mathscr{R}_d(ma, mb)$ exists, and is bounded above by $m(a+b-1)+1$.

II. (Lower Bound)

To show $\mathscr{R}_d(ma, mb) > m(a+b-1)$, we exhibit a valid 2-coloring of $[1, m(a+b-1)]$.

We treat the cases $a=1$ and $a>1$ separately. If $a=1$, the 2-coloring of [1, mb] given by

$$\Delta(x) = \left\lceil\frac{x}{m}\right\rceil \pmod 2$$

is valid, as in Case I.

Henceforth, let $a>1$ and write $b = qa + r$, where $0 < r \le a-1$. Note that $r=0$ is only possible if $a=1$ since $\gcd(a,b)=1$. We partition the interval $[1, m(a+b-1)]$ into intervals of length ma, except possibly for the last interval: $[1, ma], [ma+1, 2ma], [2ma+1, 3ma], \ldots, [qma+1, (q+1)ma], [(q+1)ma+1, (q+1)ma + m(r-1)]$. Note that the last interval exists only when $r>1$. It suffices to define the color of the integers in the interval [1, ma] since we must have $\chi(x) \ne \chi(x+ma)$ for a valid coloring.

Since $\gcd(a,b)=1$, corresponding to each $t \in [1,a]$, there is a unique pair λ_t, μ_t such that $t = \lambda_t b - \mu_t a$, $1 \le \lambda_t \le a$. Define $\chi : [1, ma] \to \{0,1\}$ by

$$\chi(x) \equiv \lambda_t + \mu_t \pmod 2, \tag{3}$$

where $\lceil \frac{x}{m} \rceil = t = \lambda_t b - \mu_t a$, $1 \le \lambda_t \le a$.

We claim that χ is a valid 2-coloring on $[1, m(a+b-1)]$ for the equations $y-x=ma$ and $y-x=mb$.

The coloring χ is a valid 2-coloring on $[1, m(a+b-1)]$ for the equation $y-x=ma$ by construction of χ. To show this is also a valid 2-coloring for the equation $y-x=mb$, we must show $\chi(x) \ne \chi(x+mb)$ for $x \in [1, m(a-1)]$.

Let $m(t-1) < x \le mt$, $1 \le t \le a-1$ and $t = \lambda_t b - \mu_t a$, $1 \le \lambda_t \le a$. Then $m(t+r-1) < x + mr \le m(t+r)$, and $t + r = (\lambda_t b - \mu_t a) + (b - qa) = (\lambda_t + 1)b - (\mu_t + q)a$. Two cases arise: (i) $t + r \le a$, and (ii) $t + r > a$.

Case (i). If $t + r \le a$, then $x + mr \le ma$, and so

$$\lambda_{t+r}b - \mu_{t+r}a = t + r = (\lambda_t b - \mu_t a) + (b - qa) = (\lambda_t + 1)b - (\mu_t + q)a. \qquad (4)$$

We may exclude the case $\lambda_t = a$ since that would imply $a \mid t$ and $1 \leq t \leq a - 1$. Therefore, $\lambda_{t+r} = \lambda_t + 1$ and $\mu_{t+r} = \mu_t + q$ by uniqueness of expression. From the construction of χ we have $\chi(x + ma) - \chi(x) \equiv 1(\text{mod } 2)$, and so $\chi(x + kma) - \chi(x) \equiv k(\text{mod } 2)$ for $1 \leq k \leq q$. In particular, from $(x + mb) - (x + mr) = qma$ we have $\chi(x + mb) - \chi(x + mr) \equiv q(\text{mod } 2)$. Therefore, from Eqn. (3)

$$\begin{aligned}\chi(x + mb) - \chi(x) &= \Big(\chi(x + mb) - \chi(x + mr)\Big) + \Big(\chi(x + mr) - \chi(x)\Big)\\ &\equiv q + \big(\lambda_{t+r} - \lambda_t\big) + \big(\mu_{t+r} - \mu_t\big) \ (\text{mod } 2)\\ &\equiv q + 1 + q \ (\text{mod } 2)\\ &\equiv 1 \ (\text{mod } 2).\end{aligned}$$

Case (ii). If $t + r > a$, then $0 < (x + mr) - ma \leq 2m(a - 1) - ma < ma$. Now

$$\lambda_{t+r-a}b - \mu_{t+r-a}a = t + r - a = (\lambda_t b - \mu_t a) + (b - qa) - a = (\lambda_t + 1)b - (\mu_t + q + 1)a. \qquad (5)$$

As in Case (i), we may exclude the case $\lambda_t = a$ since that would imply $a \mid t$ and $1 \leq t \leq a - 1$. Thus $\lambda_t < a$, we have $\lambda_{t+r-a} = \lambda_t + 1$ and $\mu_{t+r-a} = \mu_t + q + 1$ by uniqueness of expression. Arguing as in Case (i), we have from Eqn. (3)

$$\begin{aligned}\chi(x + mb) - \chi(x) &= \Big(\chi(x + mb) - \chi(x + mr - ma)\Big) + \Big(\chi(x + mr - ma) - \chi(x)\Big)\\ &\equiv (q + 1) + \big(\lambda_{t+r-a} - \lambda_t\big) + \big(\mu_{t+r-a} - \mu_t\big) \ (\text{mod } 2)\\ &\equiv (q + 1) + 1 + (q + 1) \ (\text{mod } 2)\\ &\equiv 1 \ (\text{mod } 2).\end{aligned}$$

This completes the proof. □

We claim that our construction is more efficient than the one in [7] in the following way: given some $n \in [1, m(a + b - 1) + 1]$, we show that determination of $\chi(n)$ takes time polylog$(ma + mb)$ (that is, time polynomial in $\log(ma + mb)$), while it can take time $O(ma + mb)$ to determine the value of the valid coloring constructed in [7] at n. Our time complexity is polynomial in the input values ma, mb, n, since integer k takes $O(\log k)$ bits to represent.

To show this, let $t = \left\lceil \frac{n \bmod ma}{m} \right\rceil = \lambda_t b - \mu_t a$, where $1 \leq \lambda_t \leq a$. Then, by definition of χ (and since χ is valid),

$$\chi(n) \equiv \left\lfloor \frac{n}{ma} \right\rfloor + \chi(n \bmod ma) \equiv \left\lfloor \frac{n}{ma} \right\rfloor + \lambda_t + \mu_t (\text{mod } 2).$$

Thus, to determine $\chi(n)$ computationally, one needs to determine (i) $(n \bmod ma)$ and $\left\lfloor \frac{n}{ma} \right\rfloor$ from n, (ii) t from $(n \bmod ma)$, and (iii) λ_t, μ_t from t. The first two are standard operations and can be performed in polylog$(ma + mb)$ time.

To determine λ_t, μ_t from t, use the extended Euclidean algorithm twice: first to determine m, a, b from ma, mb and then to get integers λ, μ such that $1 = \lambda b - \mu a$, which implies $t = (t\lambda)b - (t\mu)a$. Finally, determine integers q, r such that $t\lambda = qa + r$ such that $r \in [1, a]$, which gives $t = (t\lambda - qa)b - (t\mu + qb)a = rb - (t\mu + qb)a$, so that $\lambda_t = r, \mu_t = t\mu + qb$. All the involved operations can be performed in time $\mathrm{polylog}(ma + mb)$, proving our claim.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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