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CHARACTERIZATION AND ENUMERATION OF PALINDROMIC NUMBERS WHOSE SQUARES ARE ALSO PALINDROMIC

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Palindromic numbers are positive integers that remain unchanged when their decimal digits are reversed. We characterize palindromic numbers whose squares are also palindromic. We use this to determine the number of n-digit palindromic numbers whose squares are palindromic, and the number of palindromic numbers whose squares are palindromic and which are not greater than a fixed positive integer.

1. Introduction

Palindromes are words that read the same when read from left to right or from right to left. Palindromic numbers are positive integers which remain unchanged when their digits are reversed. Palindromic numbers have received much attention in recreational mathematics; see [4, pp. 6–7, 28–29]. Whereas it is easy to list all palindromic numbers, and even count their number up to a given positive integer, the same is far from true for palindromic numbers that are integral powers. In fact G. J. Simmons [3, p. 96] conjectured that there are no palindromic numbers of the form n^k for $k > 4$ and $n > 1$.

It is easy to see that there are infinitely many squares, cubes and fourth powers that are palindromic numbers. In fact, for each $n \geq 0$, $(10^n + 1)^k = \sum_{i=0}^{k} \binom{k}{i} 10^{ni}$ is palindromic for $k \in \{2, 3, 4\}$ since each of the binomial coefficients fails to exceed 9. Note that each of these numbers fails to be palindromic for $k \geq 5$ due to the fact that at least one binomial coefficient exceeds 9. On the other hand, palindromic powers with a nonpalindromic root are extremely rare. It is conjectured that the only palindromic power greater than 2 with a nonpalindromic root is 2201: $2201^3 = 10\,662\,526\,601$. Although there are many more palindromic squares with a nonpalindromic root, it is not known if there are infinitely many. The largest known palindromic square with a nonpalindromic root, discovered by Feng Yuan in January 2008 (see [1]), appears to be the 55-digit palindromic number

$$1\,886\,536\,671\,850\,530\,641\,991\,373\,196\,913\,731\,991\,460\,350\,581\,766\,356\,881,$$

which is the square of the 28-digit nonpalindromic number

$$1\,373\,512\,530\,649\,258\,635\,292\,477\,609.$$

The purpose of this article is to explore palindromic squares whose roots are also palindromic. More specifically, if n and n^2 are both palindromic, we: (i) characterize all such n (Theorem 4); (ii) give a formula for the number of all such n having a fixed number of digits (Theorem 5); and (iii) give a formula for the number of all such n which are bounded by a fixed positive integer N (Theorem 8) in Section 2.

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A table of palindromic squares with up to 49 digits may be found in [5]. Whereas the list contains all those palindromic squares with palindromic roots, it is not clear if the list is complete with respect to nonpalindromic roots, particularly for "large" numbers of digits. Keith [2] provides several tables and makes some conjectures in exploring the problem of palindromic squares and cubes, but does not provide any proofs.

2. Palindromic squares with palindromic roots

Throughout the rest of this paper, we assume n and n^2 are palindromic numbers. From the congruence $(50-n)^2 \equiv n^2 \pmod{100}$ we deduce that the number of distinct last two digits among squares are restricted to squares of the set of integers in $\{1, \dots, 25\}$. Further, these squares have distinct last two digits except for $\{10^2, 20^2\}$ and $\{5^2, 15^2, 25^2\}$, accounting for a total of 22 distinct possibilities for the last two digits of a square. A more detailed analysis of the last two digits of squares leads us to conclude that palindromic squares with palindromic roots must have an odd number of digits.

Lemma 1. *If both n and n^2 are palindromic numbers, then n^2 must have an odd number of digits.*

Proof. Suppose n and n^2 are both palindromic numbers, and n^2 has an even number of digits. Squares end in one of the following 22 two-digit numbers:

$$00, \quad \boldsymbol{e}1, \quad \boldsymbol{e}4, \quad 25, \quad \boldsymbol{o}6, \quad \boldsymbol{e}9, \tag{1}$$

where $\boldsymbol{o} \in \{1,3,5,7,9\}$ and $\boldsymbol{e} \in \{0,2,4,6,8\}$. Since n^2 is a palindrome, it cannot end in 00.

- If n^2 ends in $\boldsymbol{e}1$, n must end in 1 or 9. Since n^2 also begins with $1\boldsymbol{e}$, n must begin with 3 or 4.
- If n^2 ends in $\boldsymbol{e}4$, n must end in 2 or 8. Since n^2 also begins with $4\boldsymbol{e}$, n must begin with 6.
- If n^2 ends in 25, n must end in 5. Since n^2 also begins with 52, n must begin with 7.
- If n^2 ends in $\boldsymbol{o}6$, n must end in 4 or 6. Since n^2 also begins with $6\boldsymbol{o}$, n must begin with 7 or 8.
- If n^2 ends in $\boldsymbol{e}9$, n must end in 3 or 7. Since n^2 also begins with $9\boldsymbol{e}$, n must begin with 9.

Since n must begin and end with the same digit, each is impossible. Therefore n^2 cannot have an even number of digits. □

Let

$$n = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_k \cdot 10^k \tag{2}$$

be a $(k+1)$-digit palindromic number whose square

$$n^2 = b_0 + b_1 \cdot 10 + b_2 \cdot 10^2 + \cdots + b_{2k} \cdot 10^{2k} \tag{3}$$

is also palindromic. Here $a_i, b_i \in \{0, 1, \dots, 9\}$ for each i, each of a_0, a_k, b_0, b_{2k} is nonzero (the fact that $1 \le b_{2k} \le 9$ is a consequence of Lemma 1), and

$$\begin{aligned} a_{k-i} &= a_i \quad \text{for } i \in \{0, 1, \dots, \lfloor k/2 \rfloor\}, \\ b_{2k-i} &= b_i \quad \text{for } i \in \{0, 1, \dots, k\}. \end{aligned} \tag{4}$$

From (2),

$$n^2 = s_0 + s_1 \cdot 10 + s_2 \cdot 10^2 + \cdots + s_i \cdot 10^i + \cdots + s_{2k} \cdot 10^{2k}, \tag{5}$$

where

$$s_i = a_0 a_i + a_1 a_{i-1} + \cdots + a_i a_0 \quad \text{for } i \in \{0, 1, \ldots, 2k\}.$$

We adopt the convention $a_i = 0$ for $i > k$ when defining s_i. Hence

$$s_{2k-i} = a_{k-i} a_k + a_{k-i+1} a_{k-1} + \cdots + a_k a_{k-i} = a_i a_0 + a_{i-1} a_1 + \cdots + a_0 a_i = s_i \tag{6}$$

for $i \in \{0, 1, \ldots, k\}$ by (4). In particular, using (4) we have

$$s_k = a_0^2 + a_1^2 + \cdots + a_k^2. \tag{7}$$

The only single-digit numbers whose squares are palindromic are 1, 2 and 3. Henceforth we consider only those n in (2) with $k > 0$ and $a_0 \neq 0$. The following result is crucial in answering the first of our objectives mentioned in this section:

> Let $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$ be sets of real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_r$ and $y_1 \leq y_2 \leq \cdots \leq y_r$. The *rearrangement inequality* states that
>
> $$x_r y_1 + x_{r-1} y_2 + \cdots + x_1 y_r \leq x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_r y_{\sigma(r)} \leq x_1 y_1 + x_2 y_2 + \cdots + x_r y_r \tag{8}$$
>
> for every permutation σ of $\{1, 2, \ldots, r\}$.

In particular,

$$x_1 x_{\sigma(1)} + x_2 x_{\sigma(2)} + \cdots + x_r x_{\sigma(r)} \leq x_1^2 + x_2^2 + \cdots + x_r^2.$$

Hence

$$s_i = a_0 a_i + a_1 a_{i-1} + \cdots + a_i a_0 \leq a_0^2 + a_1^2 + \cdots + a_i^2 \leq s_k \tag{9}$$

for $i \in \{0, 1, \ldots, k\}$ by (7).

Theorem 2. *Let $k > 0$, and suppose $n = \sum_{i=0}^{k} a_i \cdot 10^i$ is a palindromic number. Then n^2 is a palindromic number if and only if $\sum_{i=0}^{k} a_i^2 \leq 9$.*

Proof. We use notation in (2), (3) and (5).

Suppose $s_k \leq 9$. Then $s_i \leq 9$ for each $i \in \{0, 1, \ldots, k\}$ by (9). Thus $b_i = s_i$ and $s_{2k-i} = s_i$ for each $i \in \{0, 1, \ldots, k\}$ by (6), so that $b_{2k-i} = s_{2k-i} = s_i = b_i$. Hence n^2 is a palindromic number.

Now suppose $s_k > 9$. Let ℓ be the least nonnegative integer for which $s_\ell > 9$. Thus $b_i = s_i$ for $i < \ell$ and $b_\ell \equiv s_\ell \pmod{10}$, $b_\ell < s_\ell$. Therefore $s_{2k-\ell} = s_\ell \geq 10$, so that $b_{2k-(\ell-1)} > s_{2k-(\ell-1)} = s_{\ell-1} = b_{\ell-1}$. Hence n^2 is not a palindromic number. ☐

Theorem 3. *Let $k > 0$. If both $n = \sum_{i=0}^{k} a_i \cdot 10^i$ and n^2 are palindromic numbers, then $a_i \in \{0, 1, 2\}$ for each $i \in \{0, 1, \ldots, k\}$.*

Proof. Let $k > 0$, and suppose both n and n^2 are palindromic numbers. Then $s_k = a_0^2 + a_1^2 + \cdots + a_k^2 \leq 9$ by (7) and Theorem 2. Since n cannot begin with a 0, $a_0 = a_k \neq 0$. Now since $k > 0$, no a_i can exceed 2. ☐

Theorem 4. *Suppose* $k > 0$. *Then* $n = \sum_{i=0}^{k} a_i \cdot 10^i$ *and* n^2 *are both palindromic numbers if and only if* $a_{k-i} = a_i$, $a_i \in \{0, 1, 2\}$ *for* $0 \le i \le \lfloor \frac{1}{2}k \rfloor$, $a_0 \ne 0$, *and*

(i) *if k is odd*, *and*

- *if* $a_0 = 1$, *then for* $1 \le i \le \frac{1}{2}(k-1)$, $a_i \in \{0, 1\}$ *and* $|\{i : a_i = 1\}| \le 3$; *or*
- *if* $a_0 = 2$, *then* $a_i = 0$ *for* $1 \le i \le \frac{1}{2}(k-1)$.

(ii) *if k is even*, *and*

- *if* $a_0 = 1$ *and* $a_{k/2} \in \{0, 1\}$, *then for* $1 \le i \le \frac{1}{2}(k-2)$, $a_i \in \{0, 1\}$ *and* $|\{i : a_i = 1\}| \le 3$; *or*
- *if* $a_0 = 1$ *and* $a_{k/2} = 2$, *then for* $1 \le i \le \frac{1}{2}(k-2)$, $a_i \in \{0, 1\}$ *and* $|\{i : a_i = 1\}| \le 1$; *or*
- *if* $a_0 = 2$, *then* $a_i = 0$ *for* $1 \le i \le \frac{1}{2}(k-2)$ *and* $a_{k/2} \in \{0, 1\}$.

Proof. Suppose $k > 0$, and both n and n^2 are palindromic numbers. Then $a_{k-i} = a_i$ and $a_i \in \{0, 1, 2\}$ for $0 \le i \le k/2$, $a_0 \ne 0$, by (4) and Theorem 3. From (4) and (7) we have

$$s_k = \begin{cases} 2(a_0^2 + a_1^2 + \cdots + a_{(k-1)/2}^2) & \text{if } k \text{ is odd,} \\ 2(a_0^2 + a_1^2 + \cdots + a_{(k-2)/2}^2) + a_{k/2}^2 & \text{if } k \text{ is even.} \end{cases} \tag{10}$$

We use Theorem 2 to characterize n.

Case (i): Suppose k is odd.

- If $a_0 = 1$, Theorem 2 applied to (10) gives $a_1^2 + \cdots + a_{(k-1)/2}^2 \le \frac{7}{2}$. So for $1 \le i \le \frac{1}{2}(k-1)$, $a_i \in \{0, 1\}$ and $|\{i : a_i = 1\}| \le 3$.
- If $a_0 = 2$, Theorem 2 applied to (10) gives $a_1^2 + \cdots + a_{(k-1)/2}^2 \le \frac{1}{2}$. Hence $a_i = 0$ for $k \in \{1, 2, \ldots, \frac{1}{2}(k-1)\}$.

Case (ii): Suppose k is even.

- If $a_0 = 1$ and $a_{k/2} \in \{0, 1\}$, Theorem 2 applied to (10) gives $a_1^2 + \cdots + a_{(k-2)/2}^2 \le \frac{7}{2}$. So for $1 \le i \le \frac{1}{2}(k-2)$, $a_i \in \{0, 1\}$ and $|\{i : a_i = 1\}| \le 3$.
- If $a_0 = 1$ and $a_{k/2} = 2$, Theorem 2 applied to (10) gives $a_1^2 + \cdots + a_{(k-2)/2}^2 \le \frac{3}{2}$. So for $1 \le i \le \frac{1}{2}(k-2)$, $a_i \in \{0, 1\}$ and $|\{i : a_i = 1\}| \le 1$.
- If $a_0 = 2$, Theorem 2 applied to (10) gives $2(a_1^2 + \cdots + a_{(k-2)/2}^2) + a_{k/2}^2 \le 1$. Hence $a_i = 0$ for $k \in \{1, 2, \ldots, \frac{1}{2}(k-2)\}$ and $a_{k/2} \in \{0, 1\}$. □

Theorem 5. *The number of d-digit palindromic numbers n such that* n^2 *is also palindromic is given by*

$$f_d = \begin{cases} \binom{d_1-1}{0} + \binom{d_1-1}{1} + \binom{d_1-1}{2} + \binom{d_1-1}{3} + 1 & \text{if } d = 2d_1, \\ 3\binom{d_1-1}{0} + 3\binom{d_1-1}{1} + 2\binom{d_1-1}{2} + 2\binom{d_1-1}{3} + 2 & \text{if } d = 2d_1 + 1. \end{cases}$$

Proof. The characterization of palindromic numbers n such that n^2 is also palindromic is given by Theorem 4. Since n is palindromic, we count the number of ordered tuples $(a_0, a_1, \ldots, a_{(k-1)/2})$ for odd k and the number of ordered tuples $(a_0, a_1, \ldots, a_{k/2})$ for even k. In all cases, $a_0 \in \{1, 2\}$ and $a_i \in \{0, 1\}$ for $i \in \{1, 2, \ldots, \lfloor k/2 \rfloor\}$ except that $a_{k/2}$ can also equal 2 for even k. Note that according to the notation in (2), n has $k + 1$ digits since $a_0 \ne 0$.

Suppose n has $d = 2d_1$ digits. Thus $k = d - 1$ is odd, and $\frac{1}{2}(k-1) = d_1 - 1$. If $a_0 = 1$, there are at most three nonzero a_i for $i \in \{1, 2, \ldots, d_1 - 1\}$. There are

$$\binom{d_1-1}{0} + \binom{d_1-1}{1} + \binom{d_1-1}{2} + \binom{d_1-1}{3}$$

such choices. There is a unique number corresponding to $a_0 = 2$.

Suppose n has $d = 2d_1 + 1$ digits. Thus $k = d - 1$ is even, and $\frac{1}{2}(k-2) = d_1 - 1$. If $a_0 = 1$ and $a_{k/2}$ is either 0 or 1, there are at most three nonzero a_i for $i \in \{1, 2, \ldots, d_1 - 1\}$. There are

$$\binom{d_1-1}{0} + \binom{d_1-1}{1} + \binom{d_1-1}{2} + \binom{d_1-1}{3}$$

such choices for each of the two choices of $a_{k/2}$. If $a_0 = 1$ and $a_{k/2} = 2$, there is at most one nonzero a_i for $i \in \{1, 2, \ldots, d_1 - 1\}$. There are

$$\binom{d_1-1}{0} + \binom{d_1-1}{1}$$

such choices for this choice of $a_{k/2}$. There are two numbers corresponding to $a_0 = 2$. □

We close this article by determining the number of palindromic numbers n that are not greater than a given positive integer N, and for which n^2 is also palindromic. Let $\boldsymbol{S}$ denote the set of palindromic numbers whose square is also palindromic, and for each positive integer N, let $\boldsymbol{f}(N) = |\{n \in \boldsymbol{S} : n \le N\}|$. If $\boldsymbol{S}_d$ is the set of d-digit numbers in $\boldsymbol{S}$, then $\boldsymbol{f}_d = |\boldsymbol{S}_d|$ is given by Theorem 4. If $\boldsymbol{f}_d(N) = |\{n \in \boldsymbol{S}_d : n \le N\}|$, then

$$\boldsymbol{f}(N) = \boldsymbol{f}_1 + \boldsymbol{f}_2 + \cdots + \boldsymbol{f}_{d-1} + \boldsymbol{f}_d(N) \tag{11}$$

for any d-digit number N.

Definition 6. The *palindromic completion* of $N = \sum_{i=0}^{d-1} A_i \cdot 10^i$ is the palindromic number $N^\star = \sum_{i=0}^{d-1} A_i^\star \cdot 10^i$, where $A_i^\star = A_i$ for $i \ge \lfloor \frac{d}{2} \rfloor$.

Definition 7. Let $N = \sum_{i=0}^{d-1} A_i \cdot 10^i$. Let $\epsilon(N)$ equal 0 if $N^\star > N$, and 1 if $N^\star \le N$. Set $s = \max\{i : A_i > 1\}$ if such an i exists, and 0 otherwise. Let $T = \{i : s < i < d-1, i \ge \lfloor \frac{d}{2} \rfloor, A_i = 1\}$. Let $r = |T|$, and write $T = \{t_1, t_2, t_3, \ldots, t_r\}$, with $t_1 > t_2 > t_3 > \cdots > t_r$, when $r > 0$.

Theorem 8. *For each $d \ge 1$, let $\boldsymbol{S}_d$ denote the set of d-digit palindromic numbers whose square is also palindromic, and let $\boldsymbol{f}_d = |\boldsymbol{S}_d|$. For each positive integer N, let $\boldsymbol{f}_d(N) = |\{n \in \boldsymbol{S}_d : n \le N\}|$.*

Let

$$N = A_0 + A_1 \cdot 10 + A_2 \cdot 10^2 + \cdots + A_{d-1} \cdot 10^{d-1}, \tag{12}$$

with $A_i \in \{0, 1, \ldots, 9\}$ for each i, $A_{d-1} \ne 0$.

(I) *Suppose $A_{d-1} > 1$. Then*

$$\boldsymbol{f}_d(N) = \begin{cases} \boldsymbol{f}_d - 2 & \text{if } d \text{ is odd and } N < 2\,0_{d-2}\,2; \\ \boldsymbol{f}_d - 1 & \text{if } d \text{ is odd and } 2\,0_{d-2}\,2 \le N < 2\,0_{(d-3)/2}\,1\,0_{(d-3)/2}\,2, \text{ or} \\ & \text{if } d \text{ is even and } N < 2\,0_{d-2}\,2; \\ \boldsymbol{f}_d & \text{otherwise.} \end{cases}$$

(II) *Suppose* $A_{d-1} = 1$.

SUBCASE (A). *Suppose* $d = 2d_1$ *is even.*

- *If* $r = 0$, *then*

$$f_d(N) = \begin{cases} \displaystyle\sum_{i=0}^{3} \binom{s-d_1+1}{i} & \text{if } s \geq d_1; \\ 1 & \text{if } s < d_1 \text{ and } N > 10_{2d_1-1}; \\ 0 & \text{otherwise.} \end{cases}$$

- *If* $1 \leq r \leq 3$, *then*

$$f_d(N) = \sum_{i=0}^{3} \binom{t_1-d_1}{i} + \sum_{i=0}^{2} \binom{t_2-d_1}{i} + \sum_{i=0}^{1} \binom{t_3-d_1}{i} + \begin{cases} \displaystyle\sum_{i=0}^{3-r} \binom{s-d_1+1}{i} & \text{if } s \geq d_1; \\ \epsilon(N) & \text{if } s < d_1. \end{cases}$$

- *If* $r > 3$, *then*

$$f_d(N) = \sum_{i=0}^{3} \binom{t_1-d_1}{i} + \sum_{i=0}^{2} \binom{t_2-d_1}{i} + \sum_{i=0}^{1} \binom{t_3-d_1}{i} + 1.$$

SUBCASE (B). *Suppose* $d = 2d_1 + 1$ *is odd.*

- *If* $r = 0$, *then*

$$f_d(N) = \begin{cases} \displaystyle 3\sum_{i=0}^{1} \binom{s-d_1}{i} + 2\sum_{i=2}^{3} \binom{s-d_1}{i} & \text{if } s \geq d_1; \\ 1 & \text{if } s < d_1 \text{ and } N > 10_{2d_1}; \\ 0 & \text{otherwise.} \end{cases}$$

- *If* $r = 1$, *then*

$$f_d(N) = 3\sum_{i=0}^{1} \binom{t_1-d_1-1}{i} + 2\sum_{i=2}^{3} \binom{t_1-d_1-1}{i} + \begin{cases} \displaystyle 2\sum_{i=0}^{2} \binom{s-d_1}{i} + 1 & \text{if } s \geq d_1; \\ \epsilon(N) & \text{if } s < d_1. \end{cases}$$

- *If* $r = 2$, *then*

$$f_d(N) = 3\sum_{i=0}^{1} \binom{t_1-d_1-1}{i} + 2\sum_{i=2}^{3} \binom{t_1-d_1-1}{i} + 2\sum_{i=0}^{2} \binom{t_2-d_1-1}{i} + 1 + \begin{cases} \displaystyle 2\sum_{i=0}^{1} \binom{s-d_1}{i} & \text{if } s \geq d_1; \\ \epsilon(N) & \text{if } s < d_1. \end{cases}$$

- *If* $r = 3$, *then*

$$\boldsymbol{f}_d(N) = 3\sum_{i=0}^{1}\binom{t_1-d_1-1}{i} + 2\sum_{i=2}^{3}\binom{t_1-d_1-1}{i} + 2\sum_{i=0}^{2}\binom{t_2-d_1-1}{i} + 2\sum_{i=0}^{1}\binom{t_3-d_1-1}{i} + 1 + \begin{cases} 2 & \text{if } s \geq d_1; \\ \epsilon(N) & \text{if } s < d_1. \end{cases}$$

- *If* $r > 3$, *then*

$$\boldsymbol{f}_d(N) = 3\sum_{i=0}^{1}\binom{t_1-d_1-1}{i} + 2\sum_{i=2}^{3}\binom{t_1-d_1-1}{i} + 2\sum_{i=0}^{2}\binom{t_2-d_1-1}{i} + 2\sum_{i=0}^{1}\binom{t_3-d_1-1}{i} + 3.$$

Remark 9. All binomial coefficients $\binom{n}{k}$ are set equal to 0 when $n < 0$ or undefined, or when $k > n$. We set $\binom{0}{0}$ equal to 1.

Proof. From Theorem 4 we know that there are at most two integers in $\boldsymbol{S}_d$ that do not begin with 1. One of these is $2\,0_{d-2}\,2$; this is the only one when d is even, but when d is odd, $2\,0_{(d-3)/2}\,1\,0_{(d-3)/2}\,2$ also belongs to $\boldsymbol{S}_d$. Suppose N is a d-digit number.

(I) If N does not begin with 1, then $\boldsymbol{f}_d(N)$ equals $\boldsymbol{f}_d$ unless one or both the numbers that begin with 2 in $\boldsymbol{S}_d$ exceed N. In the exceptional cases $\boldsymbol{f}_d(N)$ equals either 1 or 2 less than $\boldsymbol{f}_d$, as is easily verified case by case.

(II) Suppose N begins with 1. Throughout this proof, we let $n = \sum_{i=0}^{d-1} a_i \cdot 10^i$ with $a_{d-1} = 1$. We consider the case where d is odd and the case d is even separately, and use Theorem 4.

SUBCASE (A). Suppose $d = 2d_1$ is even.

If $r = 0$ and $s \geq d_1$, then $A_i = 0$ for $i > s$ and $A_s \geq 2$. Thus $n \in \boldsymbol{S}_d$ and $n \leq N$ if and only if $a_i = A_i$ for $i > s$ and $a_i \in \{0, 1\}$ for $d_1 \leq i \leq s$, with at most three of these a_i equal to 1. There are $\sum_{i=0}^{3}\binom{s-d_1+1}{i}$ such choices for the a_i. If $s < d_1$, then N begins with 10_{d_1-1}, so that $n \in \boldsymbol{S}_d$ and $n \leq N$ if and only if $n = N^\star = 10_{d-2}1$. Thus there is a unique choice for $n \in \boldsymbol{S}_d$ with $n \leq N$, unless $N = 10_{d-1}$ in which case there is no choice.

Suppose $r \geq 1$. Consider the following cases: (i) $a_i = A_i$ for $i > t_1$ and $a_{t_1} = 0$; (ii) $a_i = A_i$ for $i > t_2$ and $a_{t_2} = 0$; (iii) $a_i = A_i$ for $i > t_3$ and $a_{t_3} = 0$; (iv) $a_i = A_i$ for $i \geq t_3$.

If $r > 3$, then $n \in \boldsymbol{S}_d$ and $n \leq N$ if and only if one of (i)–(iv) holds. In (i), $a_i \in \{0, 1\}$ for $d_1 \leq i < t_1$, with at most three of these a_i equal to 1; there are $\sum_{i=0}^{3}\binom{t_1-d_1}{i}$ such choices for the a_i. In (ii), $a_i \in \{0, 1\}$ for $d_1 \leq i < t_2$, with at most two of these a_i equal to 1; there are $\sum_{i=0}^{2}\binom{t_2-d_1}{i}$ such choices for the a_i. In (iii), $a_i \in \{0, 1\}$ for $d_1 \leq i < t_3$, with at most one of these a_i equal to 1; there are $\sum_{i=0}^{1}\binom{t_3-d_1}{i}$ such choices for the a_i. In (iv), the only possibility for n is the one with $a_i = A_i$ for $i \geq t_3$ and $a_i = 0$ for $d_1 \leq i < t_3$.

If $r \leq 3$, then $n \in \boldsymbol{S}_d$ and $n \leq N$ if and only if one of the first r of (i)–(iii) holds, together with an additional condition which depends on whether $s \geq d_1$ or $s < d_1$. The first r of the conditions (i)–(iii) lead to the number of choices in the corresponding conditions, as given in case $r > 3$. If $s \geq d_1$, additionally n could satisfy the condition $a_i = A_i$ for $i > s$. Since $a_i \in \{0, 1\}$ for $s < i < d$ with exactly r of these a_i equal to 1, we must have $a_i \in \{0, 1\}$ for $d_1 \leq i \leq s$ with at most $3 - r$ of these a_i equal to 1. This accounts for the term $\sum_{i=0}^{3-r}\binom{s-d_1+1}{i}$ corresponding to this additional condition that applies if $s \ge d_1$. If $s < d_1$, then $A_i = \{0, 1\}$ for $d_1 \le i < d$, with exactly r of the A_i equal to 1. Since $A_i = 0$ for $d_1 \le i < t_r$, we must have $a_i = A_i$ for $d_1 \le i < d$. Now $n \in \boldsymbol{S}_d$ and $n \le N$ if and only if $n^\star = N^\star \le N$. This accounts for the term $\epsilon(N)$ corresponding to the additional condition when $s < d_1$.

SUBCASE (B). Suppose $d = 2d_1 + 1$ is odd.

If $r = 0$ and $s > d_1$, then $A_i = 0$ for $i > s$ and $A_s \ge 2$. Thus $n \in \boldsymbol{S}_d$ and $n \le N$ if and only if $a_i = A_i$ for $i > s$, $a_i \in \{0, 1\}$ for $d_1 < i \le s$, $a_{d_1} \in \{0, 1, 2\}$, with at most three of the a_i for $i \in [d_1 + 1, s]$ equal to 1 when $a_{d_1} \in \{0, 1\}$ and with at most one of the a_i for $i \in [d_1 + 1, s]$ equal to 1 when $a_{d_1} = 2$. There are $2\sum_{i=0}^{3}\binom{s-d_1}{i} + \sum_{i=0}^{1}\binom{s-d_1}{i}$ such choices for the a_i.

If $s = d_1$, then N begins with 10_{d_1-1} and $A_{d_1} \ge 2$. For $i \in \{0, 1, 2\}$, let N_i have the same digits as N except that at the d_1-th place N_i has i. So $N_2 = N$ if $A_{d_1} = 2$. Then $n \in \boldsymbol{S}_d$ and $n \le N$ if and only if $n = N_i^\star$ with $i \in \{0, 1, 2\}$. Thus there are 3 choices for $n \in \boldsymbol{S}_d$ with $n \le N$. Note that $2\sum_{i=0}^{3}\binom{s-d_1}{i} + \sum_{i=0}^{1}\binom{s-d_1}{i} = 3$ when $s = d_1$.

If $s < d_1$, then N begins with 10_{d_1}, so that $n \in \boldsymbol{S}_d$ and $n \le N$ if and only if $n = N^\star = 10_{d-2}1$. Thus there is a unique choice for $n \in \boldsymbol{S}_d$ with $n \le N$, unless $N = 10_{d-1}$ in which case there is no such choice.

Suppose $r \ge 1$. Consider the cases as in SUBCASE A: (i) $a_i = A_i$ for $i > t_1$ and $a_{t_1} = 0$; (ii) $a_i = A_i$ for $i > t_2$ and $a_{t_2} = 0$; (iii) $a_i = A_i$ for $i > t_3$ and $a_{t_3} = 0$; (iv) $a_i = A_i$ for $i \ge t_3$.

If $r > 3$, then $n \in \boldsymbol{S}_d$ and $n \le N$ if and only if one of (i)–(iv) holds. In (i), $a_i \in \{0, 1\}$ for $d_1 < i < t_1$, with at most three of these a_i equal to 1 when $a_{d_1} \in \{0, 1\}$ and at most one of these a_i equal to 1 when $a_{d_1} = 2$; there are $2\sum_{i=0}^{3}\binom{t_1-d_1-1}{i} + \sum_{i=0}^{1}\binom{t_1-d_1-1}{i}$ such choices for the a_i. In (ii), $a_i \in \{0, 1\}$ for $d_1 < i < t_2$, with at most two of these a_i equal to 1 when $a_{d_1} \in \{0, 1\}$ and all $a_i = 0$ when $a_{d_1} = 2$; there are $2\sum_{i=0}^{2}\binom{t_2-d_1-1}{i} + 1$ such choices for the a_i. In (iii), $a_i \in \{0, 1\}$ for $d_1 < i < t_3$, with at most one of these a_i equal to 1 when $a_{d_1} \in \{0, 1\}$ and $a_{d_1} \ne 2$; there are $2\sum_{i=0}^{1}\binom{t_3-d_1-1}{i}$ such choices for the a_i. In (iv), the only two possibilities for n are the ones with $a_i = A_i$ for $i \ge t_3$, $a_i = 0$ for $d_1 < i < t_3$ with $a_{d_1} \in \{0, 1\}$.

If $r \le 3$, then $n \in \boldsymbol{S}_d$ and $n \le N$ if and only if one of the first r of (i)–(iii) holds, together with an additional condition which depends on whether $s > d_1$ or $s = d_1$ or $s < d_1$. The first r of the conditions (i)–(iii) lead to the number of choices in the corresponding conditions, as given in case $r > 3$. We deal with the cases $r = 1, 2, 3$ separately.

If $r = 1$, the term corresponding to (i) is $2\sum_{i=0}^{3}\binom{t_1-d_1-1}{i} + \sum_{i=0}^{1}\binom{t_1-d_1-1}{i}$, as given in case $r > 3$. If $s > d_1$, then we may additionally satisfy $a_i = A_i$ for $i > s$ and $a_i \in \{0, 1\}$ for $i \in [d_1 + 1, s]$, with at most two of these a_i equal to 1 when $a_{d_1} \in \{0, 1\}$ and with all a_i equal to 0 when $a_{d_1} = 2$. This accounts for the additional term $2\sum_{i=0}^{2}\binom{s-d_1}{i} + 1$. If $s = d_1$, then there are the three additional possibilities for n given by $a_i = A_i$ for $i > d_1$ and $a_{d_1} \in \{0, 1, 2\}$. Note that $2\sum_{i=0}^{2}\binom{s-d_1}{i} + 1 = 3$ when $s = d_1$. If $s < d_1$, then the additional possibility for n must satisfy $a_i = A_i$ for $d_1 \le i < d$. Now $n \in \boldsymbol{S}_d$ and $n \le N$ if and only if $n^\star = N^\star \le N$. This accounts for the term $\epsilon(N)$ corresponding to the additional condition when $s < d_1$.

If $r = 2$, the term corresponding to (i) is $2\sum_{i=0}^{3}\binom{t_1-d_1-1}{i} + \sum_{i=0}^{1}\binom{t_1-d_1-1}{i}$ and that corresponding to (ii) is $2\sum_{i=0}^{2}\binom{t_2-d_1-1}{i} + 1$, as given in case $r > 3$. If $s > d_1$, then we may additionally satisfy $a_i = A_i$ for $i > s$, $a_i \in \{0, 1\}$ for $i \in [d_1 + 1, s]$ with at most one of these a_i equal to 1, and $a_{d_1} \in \{0, 1\}$. This accounts for the additional term $2\sum_{i=0}^{1}\binom{s-d_1}{i}$. If $s = d_1$, then there are the two additional possibilities for n given by $a_i = A_i$ for $i > d_1$ and $a_{d_1} \in \{0, 1\}$. Note that $2\sum_{i=0}^{1}\binom{s-d_1}{i} = 2$ when $s = d_1$. If $s < d_1$, then the argument in the case $r = 1$ holds, accounting for the term $\epsilon(N)$.

If $r = 3$, the term $2\sum_{i=0}^{3}\binom{t_1-d_1-1}{i} + \sum_{i=0}^{1}\binom{t_1-d_1-1}{i}$ corresponds to (i), the term $2\sum_{i=0}^{2}\binom{t_2-d_1-1}{i} + 1$ corresponds to (ii), and $2\sum_{i=0}^{1}\binom{t_3-d_1-1}{i}$ corresponds to (iii), as given in case $r > 3$. If $s \geq d_1$, there are the two additional possibilities for n given by $a_i = A_i$ for $i > d_1$ and $a_{d_1} \in \{0, 1\}$. If $s < d_1$, then the argument in the case $r = 1$ holds, accounting for the term $\epsilon(N)$. □

The tables provided in [5] appear to be the most complete listing of palindromic squares, both with palindromic and nonpalindromic root. The listing is complete with regard to palindromic roots up to 23 digits, and we utilize this to numerically support our results of Theorems 5 and 8. We take $d = 22$ and $d = 23$; for both cases $d_1 = 11$. We use the tables of palindromic squares of 43 and 45 digits.

For Theorem 5, we verify from the tables that

$$
\begin{aligned}
f_{22} &= \binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + 1 = 177,\\
f_{23} &= 3\binom{10}{0} + 3\binom{10}{1} + 2\binom{10}{2} + 2\binom{10}{3} + 2 = 365.
\end{aligned}
$$

For Theorem 8, we verify from the tables that:

d	d_1	s	t_1	t_2	t_3	t_4	N begins	$f_d(N)$
22	11	13	—	—	—	—	10_7	8
22	11	13	20	—	—	—	110_6	137
22	11	13	20	18	—	—	11010_4	163
22	11	13	20	18	15	—	11010010	165
22	11	13	20	18	15	14	11010011	165
22	11	10	—	—	—	—	10_{10}	1
22	11	10	20	—	—	—	110_9	131
22	11	10	20	18	—	—	11010_7	160
22	11	10	20	18	15	—	1101001_4	165
22	11	10	20	18	15	14	11010011_3	165

d	d_1	s	t_1	t_2	t_3	t_4	N begins	$f_d(N)$
23	11	13	—	—	—	—	10_8	11
23	11	13	20	—	—	—	1010_6	204
23	11	13	20	18	—	—	101010_4	246
23	11	13	20	18	15	—	101010010	250
23	11	13	20	18	15	14	101010011	250
23	11	10	—	—	—	—	10_{11}	1
23	11	10	20	—	—	—	1010_9	196
23	11	10	20	18	—	—	101010_7	241
23	11	10	20	18	15	—	10101001_4	249
23	11	10	20	18	15	14	101010011_3	250

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