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RESEARCH ARTICLE

Proportionally modular numerical semigroups generated by arithmetic progressions

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Abstract
A numerical semigroup is a submonoid of $\mathbb{Z}_{\geq 0}$ whose complement in $\mathbb{Z}_{\geq 0}$ is finite. For any set of positive integers a, b, c, the numerical semigroup $S(a, b, c)$ formed by the set of solutions of the inequality $ax \bmod b \leq cx$ is said to be proportionally modular. For any interval $[\alpha, \beta]$, $S\big([\alpha, \beta]\big)$ is the submonoid of $\mathbb{Z}_{\geq 0}$ obtained by intersecting the submonoid of $\mathbb{Q}_{\geq 0}$ generated by $[\alpha, \beta]$ with $\mathbb{Z}_{\geq 0}$. For the numerical semigroup S generated by a given arithmetic progression, we characterize a, b, c and α, β such that both $S(a, b, c)$ and $S\big([\alpha, \beta]\big)$ equal S.

Keywords Numerical semigroups · Diophantine inequalities · Proportionally modular

1 Introduction

A numerical semigroup S is a submonoid of $\mathbb{Z}_{\geq 0}$ whose complement $\mathbb{Z}_{\geq 0} \setminus S$ is finite. For the complement to be finite, it is necessary and sufficient that $\gcd(S) = 1$. For a given subset A of positive integers, we write

$$\langle A \rangle = \Big\{a_1x_1 + \cdots + a_kx_k : a_i \in A, x_i \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}\Big\} \bigcup \{0\}.$$

Note that $\langle A \rangle$ is a submonoid of $\mathbb{Z}_{\geq 0}$, and that $S = \langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

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For a numerical semigroup S, the complement $\mathbb{Z}_{\geq 0} \setminus S$ is denoted by $\mathrm{G}(S)$, and called the gap set of S. The largest element in $\mathrm{G}(S)$ is called the Frobenius number of S, and denoted by $\mathrm{F}(S)$. The cardinality of $\mathrm{G}(S)$ is called the genus of S, and denoted by $g(S)$.

We say that A is a set of generators of the numerical semigroup S, or that the numerical semigroup S is generated by the set A, when $S = \langle A \rangle$. Further, A is a minimal set of generators for S if A is a set of generators of S and no proper subset of A generates S. Every numerical semigroup has a unique minimal set of generators. The embedding dimension $e(S)$ of S is the size of the minimal set of generators.

Given a subset A of $\mathbb{Q}_{\geq 0}$, we denote by $\langle A \rangle$ the submonoid of $\mathbb{Q}_{\geq 0}$ generated by A:

$$\langle A \rangle = \left\{ a_1 x_1 + \cdots + a_k x_k : a_i \in A, x_i \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N} \right\} \bigcup \{0\}.$$

Then $S(A) = \langle A \rangle \cap \mathbb{Z}_{\geq 0}$ is a submonoid of $\mathbb{Z}_{\geq 0}$. Furthermore, $S(A)$ is a numerical semigroup. We say that $S(A)$ is the numerical semigroup associated to A. When $A = I$ is an interval, there is a simple way to describe $\langle A \rangle$.

Proposition 1 ([8, Lemma 1, pp. 282]) *Let $x \in \mathbb{Q}^+$ and let I be an interval. Then $x \in \langle I \rangle$ if and only if there exists a positive integer n such that $\frac{x}{n} \in I$.*

A proportionally modular Diophantine inequality is an expression of the form $ax \bmod b \leq cx$, where a, b, c are positive integers. The set of integer solutions of a proportionally modular Diophantine inequality form a numerical semigroup. For $a, b, c \in \mathbb{N}$, define

$$S(a, b, c) = \{x \in \mathbb{Z}_{\geq 0} : ax \bmod b \leq cx\}.$$

A numerical semigroup of this form is called proportionally modular. Since the inequality $ax \bmod b \leq cx$ has the same set of integer solutions as $(a \bmod b)x \bmod b \leq cx$, we may assume that $a \in \{1, \ldots, b\}$. We note that $S(b, b, c) = \mathbb{Z}_{\geq 0}$.

Toms [12] in an attempt to solve problems in classification theory in $C^\star$-algebras defined Toms decomposable numerical semigroups. Rosales et al [10] introduced the concept of proportionally modular numerical semigroups [10]. Rosales & Garcia-Sanchez [9] showed that a numerical semigroup is Toms decomposable if and only if it is the intersection of finitely many proportionally modular numerical semigroups.

Given a proportionally modular numerical semigroup, there exists a dual proportionally modular numerical semigroup.

Proposition 2 ([4, Proposition 1, pp. 416]) *Let $a, b, c \in \mathbb{N}$ with $c < a < b$. Then*

$$S(a, b, c) = S(b + c - a, b, c).$$

Triples (a', b', c') other than $(b + c - a, b, c)$ satisfy $S(a, b, c) = S(a', b', c')$, as borne out by Theorem 14.

We note that $S\big([\alpha, \beta]\big) = \mathbb{Z}_{\geq 0}$ if and only if $\alpha \leq \frac{1}{n} \leq \beta$ for some $n \in \mathbb{N}$, and that $S(a, b, c) = \mathbb{Z}_{\geq 0}$ if and only if $a \bmod b \leq c$. The following two propositions give a connection between $S(a, b, c)$ and $S\big([\alpha, \beta]\big)$.

Proposition 3 ([11, Lemma 1, pp. 454]) *Let $a, b, c \in \mathbb{N}$ with $c < a$. Then*

$$S(a, b, c) = S\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right).$$

Proposition 4 ([11, Lemma 1, pp. 454]) *Let $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ with $\frac{p}{q} < \frac{r}{s}$. Then*

$$S\left(\left[\frac{p}{q}, \frac{r}{s}\right]\right) = S(qr, pr, qr - ps).$$

Let $a, b, c \in \mathbb{N}$ with $c < a < b$. Using Propositions 2 and 3 , there exist intervals $[\alpha, \beta]$ and $[\alpha', \beta']$, with rational endpoints, and with $\alpha > 1$ and $\alpha' > 1$, such that

$$S([\alpha, \beta]) = S\left([\alpha', \beta']\right).$$

Moreover, it easily follows from Propositions 2 and 3 that

$$\alpha' = \frac{\beta}{\beta - 1}, \quad \beta' = \frac{\alpha}{\alpha - 1}. \tag{1}$$

The pair α', β' is not uniquely determined; see for instance, Theorem 12.

Bézout sequences are closely connected to the study of proportionally modular numerical semigroups; see [11]. A Bézout sequence is an increasing finite sequence of rational numbers $\left\{\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right\}$ with $\gcd(p_k, q_k) = 1$ satisfying $p_{k+1}q_k - p_k q_{k+1} = 1$ for $k \in \{1, \ldots, n-1\}$. A Bézout sequence is said to be *proper* if $p_{k+\ell}q_k - p_{k+\ell}q_k > 1$ for each k and each $\ell > 1$. As a consequence, we have

$$p_1 > \cdots > p_r < p_{r+1} < \cdots < p_n \tag{2}$$

for some $r \in \{1, \ldots, n\}$; see [11, Corollary 18, pp. 459].

From the proof of Lemma 22 and Theorem 23 in [11], given an interval $I = [\alpha, \beta]$, $\alpha, \beta \in \mathbb{Q}$ and $S = S(I)$, there exists a permutation of the generators $a_1, \ldots, a_e$ of S and positive integers $b_1, \ldots, b_e$ such that

$$\left\{\frac{a_n}{b_n}\right\}_{1 \le n \le e} \text{ forms a proper Bézout sequence, and } \alpha \le \frac{a_1}{b_1} < \frac{a_e}{b_e} \le \beta. \tag{3}$$

Given rational numbers p_1/q_1 and p_n/q_n in reduced form, there exists a unique proper Bézout sequence with first term p_1/q_1 and last term p_n/q_n. This unique Bézout sequence has been determined algorithmically; see [3, Algorithm 3.5].

The following proposition gives a characterization of proportionally modular numerical semigroups in terms of its minimal set of generators.

Proposition 5 ([11, Theorem 31, pp. 463]) *A numerical semigroup S is proportionally modular if and only if there exists a permutation $a_1, \ldots, a_e$ of its minimal set of generators such that the following two conditions hold:*

(i) $\gcd(a_k, a_{k+1}) = 1$ *for* $1 \le k \le e-1$, *and*
(ii) $a_k \mid (a_{k-1} + a_{k+1})$ *for* $2 \le k \le e-1$.

Certain aspects of the numerical semigroup generated by an arithmetic progression $a, a+d, a+2d, \ldots, a+(k-1)d$ have been studied by many authors (see [1,2,5–7,13]). In particular, the gap set $\mathrm{G}(S)$ and the Frobenius number $\mathrm{F}(S)$ are determined by the following result in [13]. It can be shown that the embedding dimension $e(S) = \min\{a, k\}$.

Proposition 6 ([13, Theorem 1, pp. 780]) *Let* a, d, k *be positive integers, with* $\gcd(a, d) = 1$ *and* $k \ge 2$. *Let* $S = \langle a, a+d, a+2d, \ldots, a+(k-1)d\rangle$. *Then*

(i) $G(S) = \left\{ax + dy : 0 \le x \le \left\lfloor \frac{y-1}{k-1} \right\rfloor, 1 \le y \le a-1\right\}$.
(ii) $F(S) = \max G(S) = a\left\lfloor \frac{a-2}{k-1} \right\rfloor + d(a-1)$.

Given a numerical semigroup S, it is natural to ask for a characterization of positive integers a, b, c for which $S(a, b, c) = S$, and also for a characterization of intervals $I = [\alpha, \beta]$ for which $S\big([\alpha, \beta]\big) = S$. We note that Propositions 2 and 3 connect the numerical semigroups $S(a, b, c)$ and $S\big([\alpha, \beta]\big)$, so characterizing one characterizes the other. Let $S = S\big(AP(a, d; k)\big) = \langle a, a+d, a+2d, \ldots, a+(k-1)d\rangle$, $k \ge 2$. Since $S\big(AP(a, d; k)\big) = S\big(AP(a, d; a)\big)$ whenever $k > a$, we may assume, without loss of generality, that $k \le a$.

In this paper, we characterize a, b, c and α, β such that both $S(a, b, c)$ and $S\big([\alpha, \beta]\big)$ equal the numerical semigroup $S = S\big(AP(a, d; k)\big)$ generated by any arithmetic progression. The two characterizations appear as Theorems 12 and 14 , and are based on Propositions 9, 10, and 11 . Delgado & Rosales [4] provide an algorithmic formula to determine $\mathrm{F}\big(S(a, b, c)\big)$ that does not lead to an explicit formula. The determination of an explicit formula for $\mathrm{F}\big(S(a, b, c)\big)$ therefore remains an open problem. Our characterization of a, b, c for which $S = S\big(AP(a, d; k)\big)$ yields an explicit formula in these special cases since the problem of determining the Frobenius number for numerical semigroups generated by arithmetic progressions is well known, as mentioned above. We note in passing that the algorithm in [3, Algorithm 3.5] to compute the unique proper Bézout sequence connecting two given reduced rational numbers p_1/q_1 and p_n/q_n relies on knowing these rational numbers. Since there is no obvious relationship between p_1/q_1 and p_n/q_n, and the numerical semigroup S, these endpoints of the proper Bézout sequence have to be determined first. This means that the algorithm in [3, Algorithm 3.5] is not directly useful in resolving the problem.

2 Main results

In this section, we consider an inverse problem related to two types of numerical semigroups that were introduced in Sect. 1 – numerical semigroups generated by intervals $[\alpha, \beta]$ and proportionally modular numerical semigroups $S(a, b, c)$. Propositions 2 and 3 in Sect. 1 connect these two types of numerical semigroups.

In order that the numerical semigroup generated by an arithmetic progression $a, a+d, a+2d, \ldots, a+(k-1)d$ be a numerical semigroup, it is necessary and sufficient that

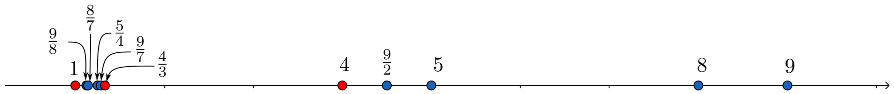

Fig. 1 For I such that $S(I) = \{0, 5, \rightarrow\}$, $[5, 9] \subset I \subset (4, \infty)$ or $[\frac{9}{2}, 8] \subset I \subset (4, \infty)$ or $[\frac{9}{7}, \frac{5}{4}] \subset I \subset (1, \frac{4}{3})$ or $[\frac{8}{7}, \frac{9}{7}] \subset I \subset (1, \frac{4}{3})$

$\gcd(a, d) = 1$. Henceforth we assume $\gcd(a, d) = 1$, and write $S\big(AP(a, d; k)\big) = \langle a, a+d, a+2d, \ldots, a+(k-1)d\rangle$. The set $\{a, a+d, a+2d, \ldots, a+(k-1)d\}$ is the minimal set of generators for $S\big(AP(a, d; k)\big)$ when $2 \le k \le a$, whereas the set $\{a, a+d, a+2d, \ldots, a+(a-1)d\}$ is the minimal set of generators for $S\big(AP(a, d; k)\big)$ for $k > a$. Throughout this section, we may therefore assume without loss of generality that $2 \le k \le a$.

We illustrate our main result, Theorem 12, with the following two examples. In Example 7 we determine intervals I for which $S(I)$ is the ordinary numerical semigroup $\{0, 5, \rightarrow\}$, and in Example 8 intervals I for which $S(I)$ is the numerical semigroup $\langle 5, 7\rangle$ with embedding dimension two. These are both special cases of our result in Theorem 12.

Example 7 We find $I = [\alpha, \beta]$, $\alpha > 1$, for which $S(I) = \{0, 5, \rightarrow\}$. From the argument leading up to equation (3), we have $\{a_1, \ldots, a_5\} = \{5, \ldots, 9\}$, giving rise to the following four Bézout sequences:

- $\frac{5}{1}, \frac{6}{1}, \frac{7}{1}, \frac{8}{1}, \frac{9}{1}$.
- $\frac{9}{2}, \frac{5}{1}, \frac{6}{1}, \frac{7}{1}, \frac{8}{1}$.
- $\frac{9}{8}, \frac{8}{7}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}$.
- $\frac{8}{7}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{9}{7}$.

This leads to

$$\alpha \le \tfrac{5}{1} < \tfrac{9}{1} \le \beta \;\text{ or }\; \alpha \le \tfrac{9}{8} < \tfrac{5}{4} \le \beta \;\text{ or }\; \alpha \le \tfrac{9}{2} < \tfrac{8}{1} \le \beta \;\text{ or }\; \alpha \le \tfrac{8}{7} < \tfrac{9}{7} \le \beta.$$

Note that the pairs of intervals $[5, 9]$, $[\frac{9}{8}, \frac{5}{4}]$ and $[\frac{9}{2}, 8]$, $[\frac{8}{7}, \frac{9}{7}]$ are connected by Eqn. (1).

From Proposition 1, $\frac{g}{n} \notin I$ whenever $g \in \mathrm{G}(S) = \{1, 2, 3, 4\}$ and $n \in \mathbb{N}$. This results in

$$\alpha \in (4, 5], \beta \in [9, \infty) \;\text{ or }\; \alpha \in \left(1, \tfrac{9}{8}\right], \beta \in \left[\tfrac{5}{4}, \tfrac{4}{3}\right) \;\text{ or }\; \alpha \in \left(4, \tfrac{9}{2}\right], \beta \in [8, \infty)$$
$$\text{or }\; \alpha \in \left(1, \tfrac{8}{7}\right], \beta \in \left[\tfrac{9}{7}, \tfrac{4}{3}\right).$$

Note that the pairs of intervals $(4, 5]$, $\left[\frac{5}{4}, \frac{4}{3}\right)$, $[9, \infty)$, $\left(1, \frac{9}{8}\right]$, $\left(4, \frac{9}{2}\right]$, $\left[\frac{9}{7}, \frac{4}{3}\right)$, and $[8, \infty)$, $\left(1, \frac{8}{7}\right]$ are connected by Eqn. (1).

By Proposition 3 and the fact that $\alpha \in (4, 5]$ and $\beta \in [9, \infty)$ we get that

$$\tfrac{b}{a} \in [4, 5),\ \tfrac{b}{a-c} \in [9, \infty) \iff \tfrac{b}{5} < a \le \tfrac{b}{4}, a - \tfrac{b}{9} \le c.$$

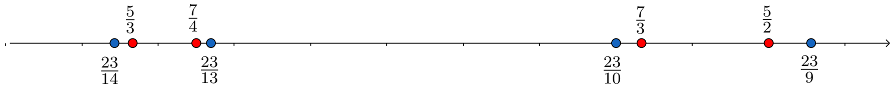

Fig. 2 For I such that $S(I) = \langle 5, 7\rangle$, $[\frac{5}{3}, \frac{7}{4}] \subset I \subset (\frac{23}{14}, \frac{23}{13})$ or $[\frac{7}{3}, \frac{5}{2}] \subset I \subset (\frac{23}{10}, \frac{23}{9})$

In the case where $\alpha \in \left(1, \frac{9}{8}\right]$ and $\beta \in \left[\frac{5}{4}, \frac{4}{3}\right)$, again using Proposition 3 we get

$$\frac{b}{a} \in \left(1, \frac{9}{8}\right], \frac{b}{a-c} \in \left[\frac{5}{4}, \frac{4}{3}\right) \iff \frac{8b}{9} \le a < b, a - \frac{4b}{5} \le c < a - \frac{3b}{4}.$$

In the other cases we get

$$\frac{b}{a} \in \left(4, \frac{9}{2}\right], \frac{b}{a-c} \in [8, \infty) \iff \frac{2b}{9} < a \le \frac{b}{4}, a - \frac{b}{8} \le c$$

and

$$\frac{b}{a} \in \left(1, \frac{8}{7}\right], \frac{b}{a-c} \in \left[\frac{9}{7}, \frac{4}{3}\right) \iff \frac{7b}{8} \le a < b, a - \frac{9b}{7} \le c < a - \frac{3b}{4}.$$

These cases together form an exhaustive list of all $a, b, c \in \mathbb{N}$ satisfying $S(a, b, c) = \{0, 5, \rightarrow\}$.

Example 8 From Proposition 5, it follows that every numerical semigroup with embedding dimension 2 is proportionally modular. We find $I = [\alpha, \beta]$, $\alpha > 1$, for which $S(I) = \langle 5, 7\rangle$. From the argument leading up to Eqn. (3), we have $\{a_1, a_2\} = \{5, 7\}$, so that $1 < \frac{5}{b_1} < \frac{7}{b_2}$ and $7b_1 - 5b_2 = 1$ or $1 < \frac{7}{b_1} < \frac{5}{b_2}$ and $5b_1 - 7b_2 = 1$. This leads to

$$\alpha \le \frac{5}{3} < \frac{7}{4} \le \beta \quad \text{or} \quad \alpha \le \frac{7}{3} < \frac{5}{2} \le \beta.$$

Note that the intervals $[\frac{5}{3}, \frac{7}{4}]$ and $[\frac{7}{3}, \frac{5}{2}]$ are connected by equation (1).

From Proposition 1, $\frac{g}{n} \notin I$ whenever $g \in \mathrm{G}(S) = \{1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23\}$ and $n \in \mathbb{N}$. This results in

$$\alpha \in \left(\frac{23}{14}, \frac{5}{3}\right], \beta \in \left[\frac{7}{4}, \frac{23}{13}\right) \quad \text{or} \quad \alpha \in \left(\frac{23}{10}, \frac{7}{3}\right], \beta \in \left[\frac{5}{2}, \frac{23}{9}\right).$$

Note that the pairs of intervals $\left(\frac{23}{14}, \frac{5}{3}\right], \left[\frac{5}{2}, \frac{23}{9}\right)$ and $\left[\frac{7}{4}, \frac{23}{13}\right), \left(\frac{23}{10}, \frac{7}{3}\right]$ are connected by Eqn. (1).

By Proposition 3, when $\alpha \in \left(\frac{23}{14}, \frac{5}{3}\right]$ and $\beta \in \left[\frac{7}{4}, \frac{23}{13}\right)$, this implies that any $a, b, c \in \mathbb{N}$ satisfying

$$\frac{3b}{5} \le a < \frac{14b}{23}, a - \frac{4b}{7} \le c < a - \frac{13b}{23}$$

will give $S(a, b, c) = \langle 5, 7\rangle$.

Similarly, when $\alpha \in \left(\frac{23}{10}, \frac{7}{3}\right]$ and $\beta \in \left[\frac{5}{2}, \frac{23}{9}\right)$ will give that any $a, b, c \in \mathbb{N}$ satisfying

$$\frac{5b}{7} \le a < \frac{10b}{23}, a - \frac{2b}{5} \le c < a - \frac{9b}{23}$$

will also give $S(a, b, c) = \langle 5, 7\rangle$.

These cases together form an exhaustive list of all $a, b, c \in \mathbb{N}$ satisfying $S(a, b, c) = \langle 5, 7\rangle$.

The main result in this section is a characterization of α, β such that $S\big([\alpha, \beta]\big) = S\big(AP(a, d; k)\big)$ in Theorem 12. Essential to our proof of Theorem 12 are three results listed as Propositions 9, 10, and 11 . We also characterize a, b, c such that $S(a, b, c) = S\big(AP(a, d; k)\big)$, via Proposition 2 and Theorem 12. We also state, without proof, the two special cases $S\big(AP(a, 1; k)\big)$ and $S\big(AP(a, 1; a)\big)$.

Proposition 9 *Let a, d, k be positive integers, with $\gcd(a, d) = 1$ and $2 \le k \le a$. Let $S\big(AP(a, d; k)\big) = \langle a, a + d, a + 2d, \ldots, a + (k-1)d\rangle$. If $dd_a = 1 + q_a a$, with $1 \le d_a < a$ and $0 \le q_a < d$, and $\lambda = \left\lfloor \frac{a-2}{k-1} \right\rfloor$, then*

$$\max\left\{\frac{g}{\left\lceil \frac{d_a g}{a}\right\rceil} : g \in G(S)\right\} = \frac{\lambda a + d(a-1)}{\lambda d_a + q_a(a-1) + 1} = \frac{F(S)}{\left\lceil \frac{d_a F(S)}{a}\right\rceil},$$

with the maximum achieved at $g = \lambda a + d(a-1)$.

Proof If $g \in \mathrm{G}(S)$, then $g = ax + dy$, with $0 \le x \le \left\lfloor \frac{y-1}{k-1}\right\rfloor$ and $1 \le y \le a-1$. Let d_a be such that $dd_a \equiv 1 \pmod{a}$, $d_a > 0$, and write $dd_a = q_a a + 1$, $q_a \in \mathbb{Z}_{\ge 0}$. Then

$$\frac{d_a g}{a} = \frac{d_a(ax + dy)}{a} = d_a x + q_a y + \frac{y}{a}.$$

Since $1 \le y < a$,

$$\frac{g}{\left\lceil \frac{d_a g}{a}\right\rceil} = \frac{ax + dy}{d_a x + q_a y + 1} \stackrel{\text{def}}{=} f_\ell(x, y).$$

We show that

$$\max\left\{f_\ell(x, y) : 0 \le x \le \left\lfloor \frac{y-1}{k-1}\right\rfloor, 1 \le y \le a-1\right\} = f_\ell\left(\left\lfloor \frac{a-2}{k-1}\right\rfloor, a-1\right) = \frac{\mathrm{F}(S)}{\left\lceil \frac{d_a \mathrm{F}(S)}{a}\right\rceil}. \tag{4}$$

Fix $y \in \{1, \ldots, a-1\}$. For $x \in \left\{0, 1, 2, \ldots, \left\lfloor \frac{y-1}{k-1} \right\rfloor - 1\right\}$, the numerator of

$$f_\ell(x+1, y) - f_\ell(x, y) = \frac{a(x+1) + dy}{d_a(x+1) + q_a y + 1} - \frac{ax + dy}{d_a x + q_a y + 1}$$

is

$$a(d_a x + q_a y + 1) - d_a(ax + dy) = a(q_a y + 1) - y(q_a a + 1) = a - y > 0.$$

Therefore

$$\begin{aligned} &\max\left\{f_\ell(x, y) : 0 \le x \le \left\lfloor \tfrac{y-1}{k-1} \right\rfloor, 1 \le y \le a-1\right\} \\ &\quad = \max\left\{f_\ell\left(\left\lfloor \tfrac{y-1}{k-1} \right\rfloor, y\right) : 1 \le y \le a-1\right\}. \end{aligned} \tag{5}$$

For $y \in \{1, \ldots, a-2\}$, the numerator of

$$f_\ell\left(\left\lfloor \tfrac{y}{k-1} \right\rfloor, y+1\right) - f_\ell\left(\left\lfloor \tfrac{y-1}{k-1} \right\rfloor, y\right) = \frac{a\left\lfloor \frac{y}{k-1} \right\rfloor + d(y+1)}{d_a\left\lfloor \frac{y}{k-1} \right\rfloor + q_a(y+1) + 1} - \frac{a\left\lfloor \frac{y-1}{k-1} \right\rfloor + dy}{d_a\left\lfloor \frac{y-1}{k-1} \right\rfloor + q_a y + 1}$$

is

$$d\left(d_a\left\lfloor \tfrac{y-1}{k-1} \right\rfloor + q_a y + 1\right) - q_a\left(a\left\lfloor \tfrac{y-1}{k-1} \right\rfloor + dy\right) = \left\lfloor \tfrac{y-1}{k-1} \right\rfloor + d > 0 \text{ if } (k-1) \nmid y,$$

and

$$\begin{aligned} &(a+d)\left(d_a\left\lfloor \tfrac{y-1}{k-1} \right\rfloor + q_a y + 1\right) - (d_a + q_a)\left(a\left\lfloor \tfrac{y-1}{k-1} \right\rfloor + dy\right) \\ &\quad = \left\lfloor \tfrac{y-1}{k-1} \right\rfloor - y + a + d > 0 \text{ if } (k-1) \mid y. \end{aligned}$$

Therefore

$$\max\left\{f_\ell\left(\left\lfloor \tfrac{y-1}{k-1} \right\rfloor, y\right) : 1 \le y \le a-1\right\} = f_\ell\left(\left\lfloor \tfrac{a-2}{k-1} \right\rfloor, a-1\right). \tag{6}$$

The claim in Eqn. (4) follows from Eqns. (5) and (6). □

Proposition 10 *Let a, d, k be positive integers, with $\gcd(a, d) = 1$ and $2 \le k \le a$. Let $S\big(AP(a, d; k)\big) = \langle a, a+d, a+2d, \ldots, a+(k-1)d\rangle$, and set $b = a + (k-1)d$. If $dd_b = 1 + q_b b$, with $1 \le d_b < b$ and $0 \le q_b < d$, and $a - 2 = \lambda(k-1) + \mu$,*

$0 \leq \mu < k-1$, *then*

$$\min\left\{\frac{g}{\left\lfloor\frac{d_b g}{b}\right\rfloor} : g \in G(S), d_b g \geq b\right\} = \frac{\lambda b + d}{\lambda d_b + q_b} = \frac{F(S) - \mu d}{\left\lfloor\frac{d_b\left(F(S)-\mu d\right)}{b}\right\rfloor},$$

with the minimum achieved at $g = \lambda a + d\big(\lambda(k-1)+1\big) = \lambda b + d$.

Proof If $g \in \mathrm{G}(S)$, then $g = ax + dy$, with $0 \leq x \leq \left\lfloor\frac{y-1}{k-1}\right\rfloor$ and $1 \leq y \leq a-1$. Let d_b be such that $dd_b \equiv 1 \pmod{b}$, $d_b > 0$, and write $dd_b = q_b b + 1$, $q_b \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}
\frac{d_b g}{b} &= \frac{d_b x\big(b-(k-1)d\big) + d_b d y}{b}\\
&= d_b x + \frac{d_b d}{b}\big(y-(k-1)x\big)\\
&= d_b x + q_b\big(y-(k-1)x\big) + \frac{y-(k-1)x}{b}.
\end{aligned}$$

Since $0 < y-(k-1)x < a < b$, we have

$$\frac{g}{\left\lfloor\frac{d_b g}{b}\right\rfloor} = \frac{ax+dy}{d_b x + q_b\big(y-(k-1)x\big)} \overset{\text{def}}{=} f_u(x,y).$$

We show that

$$\min\left\{f_u(x,y) : 0 \leq x \leq \left\lfloor\frac{y-1}{k-1}\right\rfloor, 1 \leq y \leq a-1\right\} = f_u\left(\lambda, \lambda(k-1)+1\right) = \frac{\mathrm{F}(S)-\mu d}{\left\lfloor\frac{d^{-1}\left(\mathrm{F}(S)-\mu d\right)}{b}\right\rfloor}. \tag{7}$$

Fix $y \in \{1,\ldots,a-1\}$. For $x \in \left\{0,1,2,\ldots,\left\lfloor\frac{y-1}{k-1}\right\rfloor - 1\right\}$, the numerator of

$$f_u(x+1,y) - f_u(x,y) = \frac{a(x+1)+dy}{d_b(x+1) + q_b\big(y-(k-1)(x+1)\big)} - \frac{ax+dy}{d_b x + q_b\big(y-(k-1)x\big)}$$

is

$$\begin{aligned}
a\left(d_b x + q_b\big(y-(k-1)x\big)\right) - \big(d_b - q_b(k-1)\big)(ax+dy) &= y\left(aq_b - d(d_b - q_b(k-1))\right)\\
&= y\left(q_b\big(a+(k-1)d\big) - (q_b b + 1)\right) = -y < 0.
\end{aligned}$$

Therefore

$$\min\left\{f_u(x,y) : 0 \le x \le \left\lfloor \tfrac{y-1}{k-1} \right\rfloor, 1 \le y \le a-1\right\}$$
$$= \min\left\{f_u\left(\left\lfloor \tfrac{y-1}{k-1} \right\rfloor, y\right) : 1 \le y \le a-1\right\}. \tag{8}$$

Since $y-(k-1)\left\lfloor \frac{y-1}{k-1} \right\rfloor = 1 + \big(y-1 \,(\mathrm{mod}\ k-1)\big)$, we may write

$$f_u\left(\left\lfloor \tfrac{y-1}{k-1} \right\rfloor, y\right) = \frac{a\left\lfloor \frac{y-1}{k-1} \right\rfloor + dy}{d_b\left\lfloor \frac{y-1}{k-1} \right\rfloor + q_b\left(1 + \big(y-1 \ (\mathrm{mod}\ k-1)\big)\right)}. \tag{9}$$

Write $y-1 = q(k-1)+r$, $0 \le r < k-1$. Then the minimum on the right-side of Eqn. (8) can be written as

$$\min\left\{\frac{qa+dy}{qd_b+q_b(r+1)} : 0 \le q \le \lambda-1, 0 \le r < k-1 \text{ or } q=\lambda, 0 \le r \le \mu\right\} \tag{10}$$

in view of Eqn. (9).

Fix q. For $0 \le q < \lambda$, let $0 \le r < k-1$, and for $q=\lambda$, let $0 \le r \le \mu$. The numerator of

$$f_u\left(\left\lfloor \tfrac{y}{k-1} \right\rfloor, y+1\right) - f_u\left(\left\lfloor \tfrac{y-1}{k-1} \right\rfloor, y\right)$$
$$= \frac{qa+d(y+1)}{qd_b+q_b(r+2)} - \frac{qa+dy}{qd_b+q_b(r+1)} = \frac{d\big(qd_b+q_b(r+1)\big) - q_b(qa+dy)}{\big(qd_b+q_b(r+1)\big)\big(qd_b+q_b(r+2)\big)}$$

is

$$q(q_b b - q_b a + 1) - q_b d\big(y-(r+1)\big)$$
$$= q + q\big(q_b(k-1)d\big) - q_b dq(k-1) = q > 0.$$

So for fixed q, the minimum is achieved when $r=0$, and we now have

$$\min\left\{\frac{qa+dy}{qd_b+q_b(r+1)} : 0 \le q \le \lambda-1, 0 \le r < k-1 \text{ or } q=\lambda, 0 \le r \le \mu\right\}$$
$$= \min\left\{\frac{qb+d}{qd_b+q_b} : 0 \le q \le \lambda\right\}. \tag{11}$$

Finally for $q \in \{0,1,2,\ldots,\lambda-1\}$, the numerator of

$$\frac{(q+1)b+d}{(q+1)d_b+q_b} - \frac{qb+d}{qd_b+q_b} = \frac{b(qd_b+q_b) - d_b(qb+d)}{(qd_b+q_b)\big((q+1)d_b+q_b\big)}$$

is

$$q_b b - (q_b b + 1) < 0.$$

Therefore, the minimum is achieved at $q = \lambda$, and we have

$$\min\left\{\frac{qb+d}{qd_b+q_b} : 0 \le q \le \lambda\right\} = \frac{\lambda b + d}{\lambda d_b + q_b} = f_u\left(\lambda, \lambda(k-1)+1\right). \tag{12}$$

The claim in Eqn. (7) follows from Eqns. (8) - (12). □

Proposition 11 *Let a, d be positive integers, with $\gcd(a, d) = 1$ and let $b = a + (a-1)d$. Let $S\big(AP(a, d; a)\big) = \langle a, a+d, a+2d, \ldots, a+(a-1)d\rangle$, and let*

$$dd_a = 1 + q_a a, \quad dd_b = 1 + q_b b, \quad aa_b \equiv 1 \pmod{b}, \quad dd' \equiv 1 \pmod{b-d},$$

where

$$1 \le d_a < a, \quad 0 \le q_a < d, \quad 1 \le d_b < b,$$
$$0 \le q_b < d, \quad 1 \le a_b < b, \quad 1 \le d' < b-d.$$

(i) *In terms of d_a and q_a, we have*

$$q_b = q_a, \quad d_b = d_a + (a-1)q_a,$$
$$d' = d_a + (a-2)q_a, \quad a_b = (a-1)q_a + d_a + 1.$$

(ii)

$$\left\lceil \frac{d_a g}{a} \right\rceil = \left\lceil \frac{a_b g}{b} \right\rceil \quad \textit{and} \quad \left\lfloor \frac{d_b g}{b} \right\rfloor = \left\lfloor \frac{d' g}{b-d} \right\rfloor.$$

for each $g \in G(S)$.

Proof If $g \in \mathrm{G}(S)$, then $g = dy$, with $1 \le y \le a-1$. From Propositions 9 and 10 ,

$$\left\lceil \frac{d_a g}{a} \right\rceil = q_a y + 1 \text{ and } \left\lfloor \frac{d_b g}{b} \right\rfloor = q_b y. \tag{13}$$

Multiplying both sides of $dd_a - q_a a = 1$ by $a - 1$, then adding and subtracting ad_a, and rearranging, we get

$$a\big((a-1)q_a + d_a + 1\big) - bd_a = 1.$$

Thus, $a\big((a-1)q_a + d_a + 1\big) \equiv 1 \pmod{b}$ and $1 \le (a-1)q_a + d_a + 1 < b$, so that

$$a_b = (a-1)q_a + d_a + 1. \tag{14}$$

Therefore

$$\begin{aligned}\frac{a_b g}{b} &= \frac{\big((a-1)q_a + d_a + 1\big)dy}{b}\\ &= \frac{(b-a)q_a y + \big(1+q_a a\big)y + dy}{b}\\ &= q_a y + \frac{(d+1)y}{b}.\end{aligned}$$

Since $0 \le (d+1)y \le (d+1)(a-1) < b$, we have $\left\lceil \frac{a_b g}{b} \right\rceil = q_a y + 1 = \left\lceil \frac{d_a g}{a} \right\rceil$ by Eqn. (13). This proves the first part of the proposition.

For $j \in \{0, \ldots, a-1\}$, we have $d\big(d_a + jq_a\big) - (a+jd)q_a = dd_a - q_a a = 1$. Thus, $d\big(d_a + jq_a\big) \equiv 1 \bmod (a+jd)$, and since $1 \le d_a + jq_a < a + jd$, we have $d_a + jq_a$ equals $d^{-1} \bmod (a+jd)$ for each $j \in \{0, \ldots, a-1\}$. In particular,

$$d_b = d_a + (a-1)q_a. \tag{15}$$

Multiplying both sides of Eqn. (15) by d we get

$$1 + q_b b = dd_b = d\big(d_a + (a-1)q_a\big) = (1 + q_a a) + (a-1)dq_a = 1 + q_a b.$$

Since d' is $d^{-1} \bmod a + jd$ for $j = a-2$, we have

$$d' = d_a + (a-2)q_a \;\text{ and }\; q_a = q_b. \tag{16}$$

Eqns. (14), (15) and (16) prove part (i).

We now have

$$\frac{d'g}{b-d} = \frac{\big((a-2)q_a + d_a\big)dy}{b-d} = \frac{\big((b-d)-a\big)q_a y + \big(1+q_a a\big)y}{b-d} = q_b y + \frac{y}{b-d}.$$

Since $0 \le y < b-d$, we have $\left\lfloor \frac{d'g}{b-d} \right\rfloor = q_b y = \left\lfloor \frac{d_b g}{b} \right\rfloor$ by Eqn. (13). This proves part (ii). □

Let $S = S\big(AP(a, d; k)\big)$ denote the numerical semigroup generated by AP$(a, d; k)$. Let a, b, c be any positive integers. Since $S(a, b, c) = S(a \bmod b, b, c)$ and $S(a, b, c) = \mathbb{Z}_{\ge 0}$ for $c \ge a$, we may assume without loss of generality that $c < a < b$. Propositions 3 and 4 provide a one-to-one correspondence between $S(a, b, c)$ and $S([\alpha, \beta])$. Via this correspondence, each interval $[\alpha, \beta]$, with $\alpha \ge 0$, corresponds to a unique triple (a, b, c), with $a \in \mathbb{N}$. The assumption $c < a < b$ on the positive integers a, b, c gives rise to intervals $[\alpha, \beta]$ with $\alpha > 1$. In Theorem 12, we determine all intervals $[\alpha, \beta]$ with $\alpha > 1$ such that $S([\alpha, \beta]) = S$. We then use Proposition 3 to determine all triples (a, b, c) with $c < a < b$ such that $S(a, b, c) = S$. To extend the result of Theorem 12 to all intervals $[\alpha, \beta]$ with $\alpha \ge 0$, one may combine Proposition 3 and the fact that $S(a, b, c) = S(a', b, c)$ for $a \equiv a' \pmod{b}$.

We characterize intervals $[\alpha, \beta]$, $\alpha > 1$ in Theorem 12 by calculating certain Bézout sequences related to $\mathrm{AP}(a, d; k)$. Bullejos & Rosales [3] provide an algorithm to determine the Bézout sequence connecting any two rational numbers p_1/q_1 and p_2/q_2. However, it remains to determine p_1/q_1 and p_2/q_2 in order to characterize $[\alpha, \beta]$. We give a direct method to explicitly determine the desired Bézout sequences.

Theorem 12 *Let a, d, k be positive integers, with $\gcd(a, d) = 1$, $2 \le k \le a$, $b = a + (k-1)d$, and let $\lambda = \left\lfloor \frac{a-2}{k-1} \right\rfloor$. Define $dd_a = 1 + q_a a$, where $1 \le d_a < a$ and $0 \le q_a < d$. Let $S\big(AP(a, d; k)\big) = \langle a, a+d, a+2d, \ldots, a+(k-1)d \rangle$. Then for $\alpha, \beta \in [1, \infty) \cap \mathbb{Q}$,*

$$S\big([\alpha, \beta]\big) = S\big(AP(a, d; k)\big)$$

if and only if

$$\frac{\lambda a + d(a-1)}{\lambda d_a + q_a(a-1) + 1} < \alpha \le \frac{a}{d_a}, \quad \frac{b}{d_a + (a-1)q_a} \le \beta < \frac{\lambda b + d}{\lambda\big(d_a + (a-1)q_a\big) + q_a},$$

or

$$\frac{\lambda b + d}{\lambda\big(b - d_a - (a-1)q_a\big) + (d - q_a)} < \alpha \le \frac{b}{b - d_a - (a-1)q_a},$$
$$\frac{a}{a - d_a} \le \beta < \frac{\lambda a + d(a-1)}{\lambda(a - d_a) + (d - q_a)(a-1) - 1}.$$

Moreover, for $k = a$, we also have

$$\frac{d(a-1)}{q_a(a-1) + 1} < \alpha \le \frac{b}{(a-1)q_a + d_a + 1},$$
$$\frac{b-d}{(a-2)q_a + d_a} \le \beta < \frac{d}{q_a},$$

or

$$\frac{d}{d - q_a} < \alpha \le \frac{b-d}{b - d - (a-2)q_a - d_a},$$
$$\frac{b}{b - (a-1)q_a - d_a - 1} \le \beta < \frac{d(a-1)}{(d - q_a)(a-1) - 1},$$

We adopt the convention $\frac{1}{0} = \infty$.

Proof Throughout this proof, we define

$$dd_b = 1 + q_b b, \quad aa_b \equiv 1 \pmod{b}, \quad dd' \equiv 1 \pmod{b-d},$$

with $1 \le d_b < b$, $0 \le q_b < d$, $1 \le a_b < b$, and $1 \le d' < b - d$.

Let $\alpha, \beta \in [1, \infty) \cap \mathbb{Q}$ be such that $S(I) = S(\mathrm{AP}(a, d; k)) = S$, with $I = [\alpha, \beta]$. By Proposition 1, it is enough to show that $\frac{i}{n} \notin I$ when $i \in \mathrm{G}(S)$ and $n \in \mathbb{N}$, and $\frac{j}{n} \in I$ when $j \in \mathrm{AP}(a, d; k)$ for some $n = n(j) \in \mathbb{N}$ that depends on j. By Eqn. (3), there exists a proper Bézout sequence $\left\{\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}\right\}$ such that $\alpha \le \frac{a_1}{b_1} < \frac{a_k}{b_k} \le \beta$ and $a_1, \ldots, a_k$ is a permutation of $\mathrm{AP}(a, d; k)$. Then there exists $r \in \{1, \ldots, k\}$ such that

$$a_1 > \cdots > a_r < a_{r+1} < \cdots < a_k$$

by Eqn. (2). Therefore, $a_r = a$ and $a + d = \min\{a_{r-1}, a_{r+1}\}$, unless $r = 1$ or k.

If $r = 1$, then $a_1, \ldots, a_k$ is the increasing sequence $a, a+d, \ldots, a+(k-1)d$. Thus, $a_i = a+(i-1)d$, $1 \le i \le k$. The sequence of b_i's satisfy $a_{i+1}b_i - a_i b_{i+1} = 1$, so that $b_i \equiv a_{i+1}^{-1} \pmod{a_i} \equiv d^{-1} \pmod{a_i}$ for $1 \le i \le k-1$. From $a_k b_{k-1} - a_{k-1} b_k = 1$ we have $b_k \equiv -(a_{k-1})^{-1} \pmod{a_k} \equiv d^{-1} \pmod{a_k}$. Thus, $b_i \equiv d^{-1} \pmod{a_i}$ for each $i \in \{1, \ldots, k\}$. Since $b_i, d^{-1} \pmod{a_i}$ belong to $\{1, \ldots, a_i - 1\}$, we have $b_i = d^{-1} \pmod{a_i}$, for each $i \in \{1, \ldots, k\}$. So α, β must satisfy

$$\alpha \le \frac{a_1}{b_1} = \frac{a}{d_a} < \frac{a_k}{b_k} = \frac{b}{d_b} \le \beta. \tag{17}$$

In order that $\frac{i}{n} \notin I$ whenever $i \in \mathrm{G}(S)$ and $n \in \mathbb{N}$, we must have either $\frac{i}{n} < \alpha$ or $\frac{i}{n} > \beta$. From Eqn. (17), this amounts to $\frac{i}{n} < \frac{a}{d_a}$ or $\frac{i}{n} > \frac{b}{d_b}$, or to $n > \frac{id_a}{a}$ or $n < \frac{id_b}{b}$. Therefore, α is greater than the maximum of $i/\left\lceil \frac{id_a}{a} \right\rceil$ as i runs through values in $\mathrm{G}(S)$ and β is less than the minimum of $i/\left\lfloor \frac{id_b}{b} \right\rfloor$ as i runs through values in $\mathrm{G}(S)$. Therefore, by Propositions 9 and 10

$$\alpha > \frac{\lambda a + d(a-1)}{\lambda d_a + q_a(a-1) + 1} \quad \text{and} \quad \beta < \frac{\lambda b + d}{\lambda d_b + q_b}. \tag{18}$$

This gives the first of the two ranges for α and β in the theorem using Proposition 11, part (i).

If $r = k$, then $a_1, \ldots, a_k$ is the decreasing sequence $a + (k-1)d, a + (k-2)d, \ldots, a$. Thus, $a_i = a + (k-i)d$, $1 \le i \le k$. Arguing as above, we see that $b_i = a_i - \left(d^{-1} \pmod{a_i}\right)$ for each $i \in \{1, \ldots, k\}$. So α, β must satisfy

$$\alpha \le \frac{a_1}{b_1} = \frac{b}{b - d_b} < \frac{a_k}{b_k} = \frac{a}{a - d_a} \le \beta. \tag{19}$$

In order that $\frac{i}{n} \notin I$ whenever $i \in \mathrm{G}(S)$ and $n \in \mathbb{N}$, we must have either $\frac{i}{n} < \alpha$ or $\frac{i}{n} > \beta$. From Eqn. (19), this amounts to $\frac{i}{n} < \frac{b}{b-d_b}$ or $\frac{i}{n} > \frac{a}{a-d_a}$, or to $n > \frac{i(b-d_b)}{b}$ or $n < \frac{i(a-d_a)}{a}$. Therefore, α is greater than the maximum of $i/\left\lceil \frac{i(b-d_b)}{b} \right\rceil$ as i runs through values in $\mathrm{G}(S)$ and β is less than the minimum of $i/\left\lfloor \frac{i(a-d_a)}{a} \right\rfloor$ as i runs through values in $\mathrm{G}(S)$. Since

$$\frac{m}{\left\lceil \frac{(b-d_b)m}{b} \right\rceil} < \frac{n}{\left\lceil \frac{(b-d_b)n}{b} \right\rceil} \iff \frac{n}{\left\lfloor \frac{d_b n}{b} \right\rfloor} < \frac{m}{\left\lfloor \frac{d_b m}{b} \right\rfloor}$$

and

$$\frac{m}{\left\lfloor \frac{(a-d_a)m}{a} \right\rfloor} < \frac{n}{\left\lfloor \frac{(a-d_a)n}{a} \right\rfloor} \iff \frac{n}{\left\lceil \frac{d_a n}{a} \right\rceil} < \frac{m}{\left\lceil \frac{d_a m}{a} \right\rceil}$$

for every $m, n \in \mathbb{N}$, we have

$$\alpha > \frac{\lambda b + d}{\lambda(b-d_b)+(d-q_b)} \quad \text{and} \quad \beta < \frac{\lambda a + d(a-1)}{\lambda(a-d_a)+(d-q_a)(a-1)-1}, \tag{20}$$

by Propositions 9 and 10 . This gives the second of the two ranges for α and β in the theorem using Proposition 11, part (i).

Henceforth, assume $r \neq 1, k$. Two cases arise: (i) $a_{r+1} = a + d$, and (ii) $a_{r-1} = a + d$. These cases arise only if $k = a$, as we show below.

Case (i) From $a_r \mid (a_{r-1} + a_{r+1})$ we have $a_{r-1} \equiv -a_{r+1} \equiv -d \pmod a$. If $a_{r-1} = a + id$, with $0 \leq i \leq k-1 \leq a-1$, then $i \equiv -1 \pmod a$. This is possible only if $k = a$, and then $a_{r-1} = a + (a-1)d = b$. From Eqn. (2), the sequence $a_1, \ldots, a_k$ must be $b = a + (a-1)d, a, a+d, \ldots, a + (a-2)d$. The corresponding sequence $b_2, \ldots, b_k$ is the same as in the case $r = 1$ above. Moreover, $ab_1 - \big(a + (a-1)d\big)b_2 = 1$, so that $b_1 \equiv a_b \pmod b$. Since b_1, a_b belong to $\{1, \ldots, b-1\}$, we have $b_1 = a_b$. So α, β must satisfy

$$\alpha \leq \frac{a_1}{b_1} = \frac{b}{a_b} < \frac{a_k}{b_k} = \frac{b-d}{d'} \leq \beta. \tag{21}$$

In order that $\frac{i}{n} \notin I$ whenever $i \in \mathrm{G}(S)$ and $n \in \mathbb{N}$, we must have either $\frac{i}{n} < \alpha$ or $\frac{i}{n} > \beta$. From Eqn. (21), this amounts to $\frac{i}{n} < \frac{b}{a_b}$ or $\frac{i}{n} > \frac{b-d}{d'}$, or to $n > \frac{ia_b}{b}$ or $n < \frac{id'}{b-d}$. Therefore, α is greater than the maximum of $i/\left\lceil \frac{ia_b}{b} \right\rceil$ as i runs through values in $\mathrm{G}(S)$ and β is less than the minimum of $i/\left\lfloor \frac{id'}{b-d} \right\rfloor$ as i runs through values in $\mathrm{G}(S)$. By Proposition 11, part (ii), the bounds for α and β are as given by Eqn. (18) with $\lambda = 0$. This gives the first of the two ranges for α and β in the additional case $k = a$ in the theorem using Proposition 11, part (i).

Case (ii) From $a_r \mid (a_{r-1}+a_{r+1})$ we have $a_{r+1} \equiv -a_{r-1} \equiv -d \pmod a$. Arguing as in Case (i), the sequence $a_1, \ldots, a_k$ must be $a+(a-2)d, a+(a-3)d, \ldots, a, a+(a-1)d = b$, and this is possible only if $k = a$. The corresponding sequence $b_1, \ldots, b_{k-1}$ is the same as in the case $r = k$ above, and $b_k = b - (a_b \pmod b)$. So α, β must satisfy

$$\alpha \leq \frac{a_1}{b_1} = \frac{b-d}{b-d-d'} < \frac{a_k}{b_k} = \frac{b}{b-a_b} \leq \beta. \tag{22}$$

In order that $\frac{i}{n} \notin I$ whenever $i \in \mathrm{G}(S)$ and $n \in \mathbb{N}$, we must have either $\frac{i}{n} < \alpha$ or $\frac{i}{n} > \beta$. From Eqn. (22), this amounts to $\frac{i}{n} < \frac{b-d}{b-d-d'}$ or $\frac{i}{n} > \frac{b}{b-a_b}$, or to $n > \frac{i(b-d-d')}{b-d}$ or $n < \frac{i(b-a_b)}{b}$. Therefore, α is greater than the maximum of $i/\left\lceil \frac{i(b-d-d')}{b-d} \right\rceil$ as i runs through values in $\mathrm{G}(S)$ and β is less than the minimum of $i/\left\lfloor \frac{i(b-a_b)}{b} \right\rfloor$ as i runs through values in $\mathrm{G}(S)$. Since

$$\frac{m}{\left\lceil \frac{(b-d-d')m}{b-d} \right\rceil} < \frac{n}{\left\lceil \frac{(b-d-d')n}{b-d} \right\rceil} \iff \frac{n}{\left\lfloor \frac{d'n}{b-d} \right\rfloor} < \frac{m}{\left\lfloor \frac{d'm}{b-d} \right\rfloor}$$

and

$$\frac{m}{\left\lfloor \frac{(b-a_b)m}{b} \right\rfloor} < \frac{n}{\left\lfloor \frac{(b-a_b)n}{b} \right\rfloor} \iff \frac{n}{\left\lceil \frac{a_b n}{b} \right\rceil} < \frac{m}{\left\lceil \frac{a_b m}{b} \right\rceil}$$

for every $m, n \in \mathbb{N}$, we have the bounds for α and β as given by Eqn. (20) with $\lambda = 0$ by Proposition 11, part (ii). This gives the second of the two ranges for α and β in the additional case $k = a$ in the theorem using Proposition 11, part (i). $\square$

Corollary 13 *Let a, k be positive integers, with $2 \le k \le a$, $b = a + k - 1$, and $\lambda = \left\lfloor \frac{a-2}{k-1} \right\rfloor$. Let $S\big(AP(a, 1; k)\big) = \langle a, a+1, a+2, \ldots, a+k-1 \rangle$. Then for $\alpha, \beta \in [1, \infty) \cap \mathbb{Q}$,*

$$S\big([\alpha, \beta]\big) = S\big(AP(a, 1; k)\big)$$

if and only if

$$a - \frac{1}{\lambda + 1} < \alpha \le a, \quad b \le \beta < b + \frac{1}{\lambda}$$

or

$$1 + \frac{\lambda}{\lambda(b-1)+1} < \alpha \le 1 + \frac{1}{b-1}, \quad 1 + \frac{1}{a-1} \le \beta < 1 + \frac{\lambda+1}{(\lambda+1)(a-1)-1}.$$

Moreover, for $k = a$, we also have

$$a - 1 < \alpha \le \frac{b}{2}, \quad b - 1 \le \beta,$$

or

$$1 < \alpha \le 1 + \frac{1}{b-2}, \quad 1 + \frac{2}{b-2} \le \beta < 1 + \frac{1}{a-2}.$$

We adopt the convention $\frac{1}{0} = \infty$.

Theorem 14 *Let a, d, k be positive integers, with* $\gcd(a, d) = 1$, $2 \leq k \leq a$, $b = a + (k-1)d$, *and let* $\lambda = \left\lfloor \frac{a-2}{k-1} \right\rfloor$. *Define* $dd_a = 1 + q_a a$, *where* $1 \leq d_a < a$ *and* $0 \leq q_a < d$. *Let* $S\big(AP(a, d; k)\big) = \langle a, a+d, a+2d, \ldots, a+(k-1)d\rangle$. *Then for positive integers p, q, m with $q < p < m$,*

$$S\big(p, m, q\big) = S\big(AP(a, d; k)\big)$$

if and only if

$$\frac{md_a}{a} \leq p < \frac{m\big(\lambda d_a + q_a(a-1) + 1\big)}{\lambda a + d(a-1)},$$
$$p + \frac{m\big((a-1)q_a + d_a\big)}{b} \leq q < p - \frac{m\left(\lambda\big((a-1)q_a + d_a\big) + q_a\right)}{\lambda b + d}$$

or

$$m - \frac{m\big((a-1)q_a + d_a\big)}{b} \leq p < m - \frac{m\left(\lambda\big((a-1)q_a + d_a\big) + q_a\right)}{\lambda b + d},$$
$$p - m + \frac{md_a}{a} \leq q < p - m - \frac{m\big((\lambda d_a + q_a)(a-1) - 1\big)}{\lambda a + d(a-1)}.$$

Moreover, for $k = a$, we also have

$$\frac{m\big((a-1)q_a + d_a + 1\big)}{b} \leq p < \frac{m\big(q_a(a-1) + 1\big)}{d(a-1)},$$
$$p - \frac{m\big((a-2)q_a + d_a\big)}{b-d} \leq q < p - \frac{mq_a}{d},$$

or

$$m - \frac{m\big((a-2)q_a + d_a\big)}{b-d} \leq p < m - \frac{mq_a}{d},$$
$$p - m + \frac{m\big((a-1)q_a + d_a + 1\big)}{b} \leq q < p - m + \frac{m\big(q_a(a-1) + 1\big)}{d(a-1)}.$$

Proof This follows directly from Proposition 2 and Theorem 12. If $S\big(p, m, q\big) = S\big([\alpha, \beta]\big)$, then $\frac{m}{p} = \alpha$ and $\frac{m}{p-q} = \beta$, so that $p = \frac{m}{\alpha}$ and $p - q = \frac{m}{\beta}$. The latter implies $q = p - \frac{m}{\beta}$. The bounds on α, β in Theorem 12 translate to bounds on p, q in terms of m in Theorem 14. □

Corollary 15 *Let a, k be positive integers, with $2 \leq k \leq a$, $b = a + k - 1$, and* $\lambda = \left\lfloor \frac{a-2}{k-1} \right\rfloor$. *Let* $S\big(AP(a, 1; k)\big) = \langle a, a+1, a+2, \ldots, a+k-1\rangle$. *Then for positive integers p, q, m with $q < p < m$,*

$$S\big(p, m, q\big) = S\big(AP(a, 1; k)\big)$$

if and only if

$$\frac{m}{a} \leq p < \frac{m\big(\lambda + 1\big)}{(\lambda + 1)a - 1}, \quad p + \frac{m}{b} \leq q < p - \frac{m\lambda}{\lambda b + 1}$$

or

$$m - \frac{m}{b} \leq p < m - \frac{m\lambda}{\lambda b + 1}, \quad p - m + \frac{m}{a} \leq q < p - m - \frac{m\big(\lambda(a - 1) - 1\big)}{(\lambda + 1)a - 1)}.$$

Moreover, for $k = a$, we also have

$$\frac{2m}{b} \leq p < \frac{m}{a - 1}, \quad p - \frac{m}{b - 1} \leq q < p,$$

or

$$m - \frac{m}{b - 1} \leq p < m, \quad p - m + \frac{2m}{b} \leq q < p - m + \frac{m}{a - 1}.$$

Remark 16 We note that Examples 7 and 8 follow from Corollaries 13 and 15 .

Remark 17 For any positive integer $a > 1$, for $\alpha, \beta \in [1, \infty) \cap \mathbb{Q}$, and positive integers p, q, m with $q < p < m$,

$$S\big(p, m, q\big) = S\big([\alpha, \beta]\big) = S\big(AP(a, 1; a)\big) = \big\{0, a, \rightarrow\big\}$$

is the special case $\lambda = 0$ in Corollaries 13 and 15 . We adopt the convention $\frac{1}{0} = \infty$.

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