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Some results on the spum and the integral spum of graphs

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ABSTRACT

A finite simple graph G is called a sum graph (integral sum graph) if there is a *bijection* f from the vertices of G to a set of positive integers S (a set of integers S) such that uv is an edge of G if and only if $f(u)+f(v)\in S$. For a *connected* graph G, the sum number (the integral sum number) of G, denoted by $\sigma(G)$ ($\zeta(G)$), is the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph (an integral sum graph). The spum (the integral spum) of a graph G is the minimum difference between the largest and smallest integer in any set S that corresponds to a sum graph (integral sum graph) containing G. We investigate the spum and integral spum of several classes of graphs, including complete graphs, symmetric complete bipartite graphs, star graphs, cycles, and paths. We also give sharp lower bounds for the spum and the integral spum of connected graphs.

1. Introduction

The notion of sum graph was introduced by Harary [5]. A graph $G(V,E)$ is called a sum graph if there is a *bijection* f from $V(G)$ to a set of positive integers S such that $uv\in E(G)$ for $u\neq v$ if and only if $f(u)+f(v)\in S$. We call S a set of labels for the sum graph G, and denote this set by $\mathcal{L}(G)$. Conversely, any set of positive integers S induces a sum graph G_S with vertex set S and edges s_is_j whenever $s_i+s_j\in S$. Thus every sum graph can be realized as one induced by a (finite) set of positive integers. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain at least one isolated vertex. For a *connected* graph G, the sum number of G, denoted by $\sigma(G)$, is the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph. The sum number of various classes of graphs is known, including $\mathcal{K}_n$, $\mathcal{K}_{m,n}$, $\mathcal{C}_n$ and trees; see [3, Table 20, pp. 238].

Harary [6] also generalized the notion of sum graphs by allowing the set S to contain any set of integers in the definition of sum graphs. The corresponding graph is called an integral sum graph, and the integral sum number of a connected graph G, denoted by $\zeta(G)$, is the minimum number of isolated vertices that must be added to G so that the resulting graph is an integral sum graph. Unlike sum graphs, integral sum graphs need not have isolated vertices. In fact, a conjecture of Harary [6] states that all trees have integral sum number 0. The integral sum number of a few classes of graphs is known, including $\mathcal{K}_n$ and $\mathcal{K}_{m,n}$; see [3, pp. 232].

Goodell et al. [4] investigated the difference between the largest and smallest labels in a sum graph G, and called the minimum possible such difference the spum of G. They proved the spum of $\mathcal{K}_n$ is $4n-6$, and the spum of $\mathcal{C}_n$ is at most $4n-10$, but their work seems to be unpublished [3, pp. 230]. We confirm their result on the spum of $\mathcal{K}_n$ and show that

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Table 1
Summary of results on spum and integral spum of various classes of graphs.

G	spum G	integral spum G
G	$\geq 2n-(\Delta-\delta)-2$ (Theorem 2.1)	$\geq 2n-\Delta-3$ (Theorem 2.2)
$\mathcal{K}_n$	$4n-6$ for $n \geq 2$ (Theorems 3.1 & 3.2)	$4n-6$ for $n \geq 4$ (Theorems 3.1 & 3.2)
$\mathcal{K}_{1,n}$	$2n-1$ for $n \geq 2$ (Theorems 4.1 & 4.2)	$2n-2$ for $n \geq 2$ (Theorems 4.3 & 4.4)
$\mathcal{K}_{n,n}$	$7n-7$ for $n \geq 2$ (Theorems 5.1 & 5.2)	$7n-7$ for $n \geq 2$ (Theorems 5.1 & 5.2)
$\mathcal{C}_n$	$[2n-2, 2n-1]$ for $n \geq 4$ (Remark 6.1) $2n-1$ for $n \geq 13$ (Theorems 6.1 & 6.2)	$\geq 2n-5$ (Theorem 2.2)
$\mathcal{P}_n$	$[2n-3, 2n+1]$ for $n \geq 9$, n odd $[2n-3, 2n+2]$ for $n \geq 9$, n even (Theorems 7.1 & 7.2)	$\left[2n-5, \frac{5}{2}(n-3)\right]$ for $n \geq 7$, n odd $[2n-5, 2n-3]$ for $n \geq 7$, n even (Theorems 7.3 & 7.4)

the spum of $\mathcal{C}_n$ is either $2n-1$ or $2n-2$, with the former value the answer when $n \geq 13$. We also show that the spum of $\mathcal{K}_{1,n}$ is $2n-1$, the spum of $\mathcal{K}_{n,n}$ is $7n-7$, and that the spum of $\mathcal{P}_n$ lies between $2n-3$ and $2n+2$. We obtain the sharp lower bound $2n-(\Delta-\delta)-2$ for the spum of a graph of order n and maximum and minimum vertex degree Δ and δ, respectively.

We introduce the notion of integral spum of a graph G, replacing a sum graph by an integral sum graph. We show that the integral spum of $\mathcal{K}_n$ equals $4n-6$, that of $\mathcal{K}_{1,n}$ equals $2n-2$, and that of $\mathcal{K}_{n,n}$ equals $7n-7$. We also show that the integral spum of $\mathcal{P}_n$ lies between $2n-5$ and $2n-3$ when n is even, and between $2n-5$ and $\frac{5}{2}(n-3)$ for odd n. We obtain a sharp lower bound of $2n-\Delta-3$ for the integral spum of a graph of order n and maximum vertex degree Δ. A summary of our results is given in Table 1.

Melnikov & Pyatkin [8] showed that all 2-regular graphs with the exception of $\mathcal{C}_4$ are integral sum graphs, and that for every positive integer r there exists an r-regular integral sum graph. They also introduced the notion of the integral radius $r(G)$ for integral sum graphs G. The integral radius of G is the least positive integer r for which there is an integral sum labelling L of G with $L \subseteq [-r, r]$. We remark that our results on the integral spum of graphs automatically provide bounds for the integral radius of the classes of graphs mentioned in Table 1.

Throughout this paper, X and Y are sets of integers, $X \setminus Y := \{x : x \in X, x \notin Y\}$ and $X - a := \{x - a : x \in X\}$. We denote by $\mathcal{G}$ a connected graph whose spum and integral spum we study, and by G the sum (integral sum) graph consisting of $\mathcal{G}$ and $\sigma(\mathcal{G})$ ($\zeta(\mathcal{G})$) isolated vertices. By $\mathcal{L}(G)$ we mean a labelling on the vertices of the graph G, so that the spum $\mathcal{G}$ (integral spum $\mathcal{G}$) equals $\max \mathcal{L}(G) - \min \mathcal{L}(G)$. Upper bounds for spum $\mathcal{G}$ (integral spum $\mathcal{G}$) are achieved via any sum (integral sum) labelling of the vertices of G. Sharp upper bounds require an optimal labelling of the vertices. Lower bounds for spum $\mathcal{G}$ (integral spum $\mathcal{G}$) are the difference between any maximum and minimum labellings. More specifically, if $\{a_1, \ldots, a_n\}$ is a sum or integral sum labelling arranged in increasing order, a recurrent theme in achieving lower bounds is to partition the interval of integers in $[a_1, a_n]$ into two sets, and consider in addition translates of one or the other, and the complement of the labelling in the interval. An estimate of the size of each of these sets provides us with a lower bound. We attempt to minimize the difference between the upper and lower bounds in each of these labellings, and succeed in making this difference zero in the cases of $\mathcal{K}_n$, $\mathcal{K}_{1,n}$ and $\mathcal{K}_{n,n}$, and $\mathcal{C}_n$ for $n \geq 13$.

2. Lower bounds for spum & integral spum of graphs

In this section, we give lower bounds for the spum and for the integral spum of graphs of order n in terms of their maximum and minimum vertex degrees. We show that these bounds are sharp by providing an infinite class of graphs that achieve these bounds.

Theorem 2.1. *For graphs $\mathcal{G}$ of order n, with maximum and minimum vertex degree Δ and δ respectively,*

$$\text{spum } \mathcal{G} \geq 2n - (\Delta - \delta) - 2.$$

Moreover, there exists an infinite class of graphs that achieves this bound.

Proof. Let $\mathcal{G}$ be a graph with n vertices and with maximum vertex degree $\Delta(\mathcal{G}) = \Delta$ and minimum vertex degree $\delta(\mathcal{G}) = \delta$. Let G be a sum graph consisting of $\mathcal{G}$ together with $\sigma(\mathcal{G})$ isolated vertices. Let $v_1, \ldots, v_n$ be the vertices of $\mathcal{G}$. Let L be a labelling of G for which $\max L - \min L = \text{spum } \mathcal{G}$. Let $S = \{a_1, \ldots, a_n\}$, written in increasing order, be the labelling of $\mathcal{G}$. Let $\ell(v_i) = a_i$ for $1 \leq i \leq n$. Let

$$S_1 = S \cap [a_1, 2a_1], \quad S_2 = S \cap [2a_1 + 1, a_n], \quad S_3 = S_2 - a_1, \quad T = [a_1, a_n] \setminus S.$$

We claim that $\left|S\cap S_3\right|\le \Delta$. Indeed, if $a_{i_1},\ldots,a_{i_{\Delta+1}}\in S\cap S_3$, then $a_1+a_{i_k}\in S$ for $1\le k\le \Delta+1$. Hence v_1 is adjacent to each of the vertices $v_{i_1},\ldots,v_{i_{\Delta+1}}$, so that $d(v_1)\ge \Delta+1$. This proves the claim.

Since $S_1\subseteq[a_1,2a_1]$, we have $\left|S_3\right|=\left|S_2\right|=\left|S\right|-\left|S_1\right|\ge n-(a_1+1)$. From $S_3\subset[a_1,a_n]$ we have

$$n-(a_1+1)\le\left|S_3\right|=\left|S_3\cap[a_1,a_n]\right|=\left|S_3\cap S\right|+\left|S_3\cap T\right|. \tag{1}$$

If $\left|S\cap S_3\right|=\Delta$, Eq. (1) reduces to $(a_n-a_1+1)-n=|T|\ge\left|S_3\cap T\right|\ge n-a_1-\Delta-1$, so that $a_n\ge 2n-2-\Delta$. Since $\left|S\cap S_3\right|=\Delta$, all neighbours of v_1 have labels $<a_n$. Hence v_n is not adjacent to v_1, so that $\max L-\min L\ge(a_n+a_{\delta+1})-a_1\ge a_n+\delta\ge 2n-(\Delta-\delta)-2$.

Otherwise $\left|S\cap S_3\right|\le\Delta-1$, and Eq. (1) reduces to $(a_n-a_1+1)-n=|T|\ge\left|S_3\cap T\right|\ge n-a_1-\Delta$, so that $a_n\ge 2n-1-\Delta$. Hence $\max L-\min L\ge(a_n+a_\delta)-a_1\ge a_n+\delta-1\ge 2n-(\Delta-\delta)-2$.

To show the lower bound is achieved for an infinite class of graphs, consider the sum graph induced by the labelling $\left\{\lfloor\frac{1}{2}(n-1)\rfloor,\ldots,\lfloor\frac{1}{2}(3n-3)\rfloor\right\}\cup\left\{n+2\lfloor\frac{1}{2}(n-1)\rfloor\right\}$. The vertex labelled $n+2\lfloor\frac{1}{2}(n-1)\rfloor$ is the only isolated vertex, and the graph $\mathscr{G}$ that results on removing this vertex is connected. It is easy to see that the vertex labelled $\lfloor\frac{1}{2}(n-1)\rfloor$ achieves the maximum degree $\Delta=\lfloor\frac{n}{2}\rfloor$, the vertex labelled $\lfloor\frac{1}{2}(3n-3)\rfloor$ achieves the minimum degree $\delta=1$ in this connected graph $\mathscr{G}$, and $\texttt{spum}\,\mathscr{G}=n+\lfloor\frac{1}{2}(n-1)\rfloor=2n-(\Delta-\delta)-2$. ■

Theorem 2.2. *For graphs $\mathscr{G}$ of order n with maximum vertex degree Δ,*

$$\texttt{integral spum}\;\mathscr{G}\ge 2n-\Delta-3.$$

Moreover, there exists an infinite class of graphs that achieves this bound.

Proof. Let $\mathscr{G}$ be a graph with n vertices, maximum vertex degree $\Delta(\mathscr{G})=\Delta$ and minimum vertex degree $\delta(\mathscr{G})=\delta$. Let G be an integral sum graph consisting of $\mathscr{G}$ together with $\zeta(\mathscr{G})$ isolated vertices. Let $v_1,\ldots,v_n$ be the vertices of G corresponding to the vertices of $\mathscr{G}$. Let L be a labelling of G for which $\max L-\min L=\texttt{integral spum}\,\mathscr{G}$. Let $S=\{a_1,\ldots,a_n\}$, written in increasing order, be the labelling of $\mathscr{G}$. Let $\ell(v_i)=a_i$ for $1\le i\le n$. If each integer in S has the same sign, we may assume that $a_1,\ldots,a_n$ are all positive by replacing L by $-L$ if necessary. Now the result of Theorem 2.1 applies, and so we may henceforth assume that $a_1<0<a_n$. Suppose r is such that $a_r<0<a_{r+1}$. By replacing L by $-L$ if necessary, we may further assume that $a_{r+1}\le|a_r|=-a_r$.

Let

$$S_1=\{a_1,\ldots,a_r\},\quad S_2=\{a_{r+1},\ldots,a_n\},\quad S_3=S_1+a_{r+1},\quad S_4=S_2-a_{r+1}.$$

We claim that $\left|S_1\cap S_3\right|+\left|S_2\cap S_4\right|\le\Delta+1$. This upper bound can be improved to Δ provided $2a_{r+1}\notin S_2$. If $S_1\cap S_3\ne\emptyset$, then $a_i=a_j+a_{r+1}$ with $i,j\in\{1,\ldots,r\}$. If $S_2\cap S_4\ne\emptyset$, then $a_i=a_j-a_{r+1}$ with $i,j\in\{r+1,\ldots,n\}$. However, $2a_{r+1}\in S_2$ contributes one to the count in $S_2\cap S_4$ but not to $d(v_{r+1})$. Since $d(v_{r+1})\le\Delta$, the claim follows.

Observe that each of the sets S_1, S_2, S_3, S_4 lie within the interval $[a_1,a_n]$. Moreover $S_i\cap S_j=\emptyset$ for $i\ne j$, except possibly for $(i,j)\in\{(1,3),(2,4),(3,4)\}$. Note that $\left|S_3\cap S_4\right|\le 1$, with equality if and only if $a_r+a_{r+1}=0$. Since $\left|S_1\cap S_3\right|+\left|S_2\cap S_4\right|\le\Delta+1$, we have $\max L-\min L\ge\left|\bigcup_{i=1}^4 S_i\right|-1\ge\sum_{i=1}^4\left|S_i\right|-(\Delta+2)-1=2n-\Delta-3$.

To show the lower bound is achieved for an infinite class of graphs, consider the integral sum graph $\mathscr{G}$ induced by the labelling $\left\{-\lfloor\frac{1}{2}n\rfloor,\ldots,\lfloor\frac{1}{2}(n+1)\rfloor\right\}\setminus\{0\}$. It is easy to see that the vertices labelled -1 and 1 achieve the maximum degree $\Delta=n-3$, and $\texttt{integral spum}\,\mathscr{G}=n=2n-\Delta-3$. ■

3. Complete graphs

The study of spum was initiated by Goodell et al. [4] with the calculation of spum of complete graphs and of cycles. Although they determined the spum of complete graphs and found an upper bound for cycles, their paper was unpublished. In this section we determine the spum and the integral spum of complete graphs. Bergstrand et al. [1] showed that the sum number $\sigma(\mathcal{K}_n)=2n-3$ for $n\ge 4$. It is known and easy to see that $\sigma(\mathcal{K}_2)=1$ and $\sigma(\mathcal{K}_3)=2$; see [3, Table 20, pp. 238]. Chen [2], Sharary [9], and Xu [11] proved a conjecture of Harary that the integral sum number $\zeta(\mathcal{K}_n)=2n-3$ for $n\ge 4$. It is known and easy to see that $\zeta(\mathcal{K}_2)=0$ (see [10]). We note that the labelling $\{-1,0,1\}$ shows that $\zeta(\mathcal{K}_3)=0$. Our main result is that for $n\ge 4$,

$$\texttt{spum}\,\mathcal{K}_n=4n-6,\qquad\texttt{integral spum}\,\mathcal{K}_n=4n-6.$$

We begin by determining spum and integral spum for complete graphs of order 2 and 3.

Lemma 3.1.

$$\texttt{spum}\,\mathcal{K}_2=2,\quad\texttt{integral spum}\,\mathcal{K}_2=1;$$
$$\texttt{spum}\,\mathcal{K}_3=6,\quad\texttt{integral spum}\,\mathcal{K}_3=2.$$

Proof. For $\mathtt{spum}$ of $\mathcal{K}_2$, any sum graph consisting of $\mathcal{K}_2$ together with an isolated vertex is a graph with three vertices. Hence the difference between the largest and smallest labels is at least 2. This difference is achieved by the labelling $\{1, 2, 3\}$. This proves $\mathtt{spum}\,\mathcal{K}_2 = 2$.

For $\mathtt{integral\ spum}$ of $\mathcal{K}_2$, any integral sum graph consisting of $\mathcal{K}_2$ is a graph with two vertices, so that difference between the two labels is at least 1. This difference is achieved by the labelling $\{0, 1\}$. This proves $\mathtt{integral\ spum}\,\mathcal{K}_2 = 1$.

For $\mathtt{spum}$ of $\mathcal{K}_3$, consider any sum graph consisting of $\mathcal{K}_3$ together with two isolated vertices. If the labels of the vertices of $\mathcal{K}_3$ are $\{a, b, c\}$ with $1 \le a < b < c$, then the labels of the two isolated vertices must be $c + a$ and $c + b$, and $a + b$ must equal c. Hence the difference between the maximum and minimum labels is at least $c + b - a = 2b$. We observe that $b > 2$, since $b = 2$ forces the labels $\{1, 2, 3, 4, 5\}$ and this is not the label of $\mathcal{K}_3$ together with two isolated vertices. Therefore $\mathtt{spum}\,\mathcal{K}_3 \ge 6$. This difference is achieved by the labelling $\{1, 3, 4, 5, 7\}$. This proves $\mathtt{spum}\,\mathcal{K}_3 = 6$.

For $\mathtt{integral\ spum}$ of $\mathcal{K}_3$, any integral sum graph consisting of $\mathcal{K}_3$ is a graph with three vertices, so that difference between the maximum and minimum labels is at least 2. This difference is achieved by the labelling $\{-1, 0, 1\}$. This proves $\mathtt{integral\ spum}\,\mathcal{K}_3 = 2$. ■

Theorem 3.1. *For $n \ge 4$,*

$$\mathtt{spum}\,\mathcal{K}_n \le 4n - 6, \qquad \mathtt{integral\ spum}\,\mathcal{K}_n \le 4n - 6.$$

Proof. Let G be the graph induced by the set of labels $L = [2n-3, 3n-4] \cup [4n-5, 6n-9]$. We show that the n vertices with labels in $[2n-3, 3n-4]$ form a clique while the $2n-3$ vertices with labels in $[4n-5, 6n-9]$ are isolated. To prove our claim, let $I_1 = [2n-3, 3n-4]$ and $I_2 = [4n-5, 6n-9]$. Then the sum of any two distinct integers in I_1 lies in I_2, whereas the sum of any two integers at least one of which is in I_2 lies outside $I_1 \cup I_2$.

Since $\zeta(\mathcal{K}_n) = \sigma(\mathcal{K}_n) = 2n - 3$, $\mathtt{integral\ spum}\,\mathcal{K}_n \le \mathtt{spum}\,\mathcal{K}_n \le (6n-9) - (2n-3) = 4n - 6$. ■

Theorem 3.2. *For $n \ge 4$,*

$$\mathtt{spum}\,\mathcal{K}_n \ge 4n - 6, \qquad \mathtt{integral\ spum}\,\mathcal{K}_n \ge 4n - 6.$$

Proof. Since $\sigma(\mathcal{K}_n) = \zeta(\mathcal{K}_n) = 2n - 3$ and $\mathtt{integral\ spum}$ of a graph is always bounded above by its $\mathtt{spum}$, it suffices to prove that $\mathtt{integral\ spum}\,\mathcal{K}_n \ge 4n - 6$.

Let G be an integral sum graph consisting of $\mathcal{K}_n$ together with $2n - 3$ isolated vertices. Let L be a labelling of G for which $\max L - \min L = \mathtt{integral\ spum}\,\mathcal{K}_n$. Let $S = \{a_1, \ldots, a_n\}$, written in increasing order, denote the labelling within L of the vertices corresponding to $\mathcal{K}_n$. Note that $0 \notin L$ since G has isolated vertices. Without loss of generality, we may assume that $a_n > 0$ since we may replace L by $-L$ if $a_n < 0$.

Chen [2] showed that S is sum-free. Hence the $2n - 3$ isolated vertices of G have labels in $L \setminus S$, which we partition into disjoint sets A and B as follows:

$$A = \{a_1 + a_i : 2 \le i \le n\} = a_1 + \big(S \setminus \{a_1\}\big), \quad B = \{a_n + a_i : 2 \le i \le n-1\} = a_n + \big(S \setminus \{a_1, a_n\}\big).$$

Note that $\max A = a_1 + a_n < a_2 + a_n = \min B$. Hence $\left|A \cup B\right| = \left|A\right| + \left|B\right| = (n-1) + (n-2) = 2n - 3$. Therefore, L is the disjoint union of S, A, and B.

We claim that $\big(B - |a_1|\big) \cap L = \emptyset$. We prove this separately for the cases $a_1 > 0$ and $a_1 < 0$.

Case (i) ($a_1 > 0$)

Note that $(B - a_1) \cap S = \emptyset$ since $\max S = a_n < a_n + (a_2 - a_1) = \min(B - a_1)$ and that $(B - a_1) \cap B = \emptyset$ since S is sum-free. Suppose $(B - a_1) \cap A \ne \emptyset$. Then $a_n + a_i - a_1 = a_1 + a_j$ for $2 \le i \le n-1$ and $2 \le j \le n$. But then the isolated vertex with label $a_1 + a_j$ is adjacent to the vertex with label a_1, and this is a contradiction. Hence $(B - a_1) \cap A = \emptyset$.

Case (ii) ($a_1 < 0$)

Note that since S is sum-free, $(B + a_1) \cap A = \emptyset = (B + a_1) \cap B$. Suppose $(B + a_1) \cap S \ne \emptyset$. Then $a_n + a_i + a_1 = a_j$ with $2 \le i \le n-1$ and $2 \le j \le n$. But then the isolated vertex with label $a_n + a_i$ is adjacent to the vertex with label a_1, and this is a contradiction. Hence $(B + a_1) \cap S = \emptyset$.

We show that $B - |a_1|$ is contained in the interval $[\min L, \max L]$. If $a_1 > 0$, then $\min L = \min S < \max S < \min(B - a_1) < \max(B - a_1) < \min B < \max B = \max L$. If $a_1 < 0$, then $\min L = \min\{a_1, a_1 + a_2\} \le a_1 + a_2 < a_1 + a_2 + a_n = \min\big(B + a_1\big) \le \max\big(B + a_1\big) = a_n + a_{n-1} + a_1 < a_n + a_{n-1} = \max B \le \max L$.

Since $\left|S\right| = n$, $\left|A\right| = n - 1$, $\left|B\right| = \left|B - |a_1|\right| = n - 2$, we must have $\max L - \min L \ge \left|L\right| + \left|B - |a_1|\right| - 1 = \left|S\right| + \left|A\right| + \left|B\right| + \left|B - |a_1|\right| - 1 = 4n - 6$. ■

4. Star graphs

Harary [5] showed that the sum number $\sigma(\mathcal{K}_{1,n}) = 1$ for $n \ge 2$, and that the integral sum number $\zeta(\mathcal{K}_{1,n}) = 0$ for $n \ge 2$. We show that, for $n \ge 2$,

$$\mathtt{spum}\,\mathcal{K}_{1,n} = 2n - 1, \qquad \mathtt{integral\ spum}\,\mathcal{K}_{1,n} = 2n - 2.$$

Theorem 4.1. *For $n \geq 2$,* $\texttt{spum}\ \mathcal{K}_{1,n} \leq 2n-1$.

Proof. Let G be the graph induced by the set of labels $L = \{1\} \cup [n, 2n]$. Then the vertex labelled 1 is adjacent to each of the vertices with labels in $[n, 2n-1]$, and there is no other edge. The vertex with label $2n$ is isolated. Hence G is the union of $\mathcal{K}_{1,n}$ and an isolated vertex, so that $\texttt{spum}\ \mathcal{K}_{1,n} \leq 2n-1$. ■

Theorem 4.2. *For $n \geq 2$,* $\texttt{spum}\ \mathcal{K}_{1,n} \geq 2n-1$.

Proof. Let G be a sum graph consisting of $\mathcal{K}_{1,n}$ together with an isolated vertex. Let L be a labelling of G for which $\max L - \min L = \texttt{spum}\ \mathcal{K}_{1,n}$. Let $S = \{a_1, \ldots, a_n\} \cup \{a\}$ denote the labelling within L of the vertices corresponding to $\mathcal{K}_{1,n}$, where the vertex with degree n is labelled a, and let $a_1 < \cdots < a_n$. Let $L \setminus S = \{b\}$. Since $a+a_1, \ldots, a+a_n$ forms a strictly increasing sequence of integers in $L \setminus \{a, a_1\}$, we must have $(a+a_1, \ldots, a+a_n) = (a_2, \ldots, a_n, b)$ as ordered n-tuples. Therefore $a_i - a_{i-1} = a$ for $2 \leq i \leq n$ and $b = a + a_n$, so that $L = \{a, a_1, a_1+a, \ldots, a_1+na\}$. If $a = 1$, $a_1 + (a_1+1) > a_1 + n$ since the vertices labelled a_1 and a_1+1 are not adjacent. Hence $a_1 \geq n$, and $\max L - \min L = a_1 + n - 1 \geq 2n-1$. If $a \geq 2$, then $\max L - \min L \geq a_1 + (n-1)a \geq 2(n-1)+1 = 2n-1$. ■

Theorem 4.3. *For $n \geq 2$,* $\texttt{integral spum}\ \mathcal{K}_{1,n} \leq 2n-2$.

Proof. Let G be the graph induced by the set of labels $L = \{0\} \cup [n-1, 2n-2]$. Then the vertex labelled 0 is adjacent to each of the vertices with labels in $[n-1, 2n-2]$, and there is no other edge. Hence $G \cong \mathcal{K}_{1,n}$, so that integral spum $\mathcal{K}_{1,n} \leq 2n-2$. ■

Theorem 4.4. *For $n \geq 1$,* $\texttt{integral spum}\ \mathcal{K}_{1,n} \geq 2n-2$.

Proof. Let $L = \{a_1, \ldots, a_n\} \cup \{a\}$ be a labelling of $\mathcal{K}_{1,n}$, where the vertex with degree n is labelled a and $a_1 < \cdots < a_n$, and for which $\max L - \min L = \texttt{integral spum}\ \mathcal{K}_{1,n}$. Without loss of generality, we may assume that $a \geq 0$ by replacing L by $-L$ if necessary. Since $a+a_1, \ldots, a+a_n$ forms an increasing sequence of integers in L, we must have $a + a_n \in \{a, a_n\}$. If $a + a_n = a$, then $a_n = 0$, so that a_n corresponds to a vertex of degree n. This contradicts the assumption that the vertex labelled a has degree n. Thus $a + a_n = a_n$, and $a = 0$. Hence $L = S \cup \{0\}$, where $S = \{a_1, \ldots, a_n\}$ is sum-free. Moreover a_1 must be positive; otherwise, replacing a_1 by $-a_1$ again results in a sum-free set and a valid labelling for G but with a smaller integral spum. The same argument extends to all other negative integers in S. Therefore, we may henceforth assume that $a_1 > 0$. Let

$$S_1 = S \cap [a_1, 2a_1], \quad S_2 = S \cap [2a_1+1, a_n], \quad S_3 = S_2 - a_1, \quad T = [a_1, a_n] \setminus S.$$

Note that $S \cap S_3 = \emptyset$, since S is sum-free. Since $S_1 \subseteq [a_1, 2a_1]$, $|S_3| = |S_2| = |S| - |S_1| \geq n - (a_1+1)$. From $S_3 \subset [a_1, a_n]$, we have

$$n - (a_1+1) \leq |S_3| = |S_3 \cap [a_1, a_n]| = |S_3 \cap S| + |S_3 \cap T|. \tag{2}$$

Since $|S \cap S_3| = 0$, Eq. (2) reduces to $(a_n - a_1 + 1) - n = |T| \geq |S_3 \cap T| \geq n - a_1 - 1$, so that $a_n \geq 2n-2$. Hence $\max L - \min L \geq 2n-2$. ■

5. Complete symmetric bipartite graphs

In this section we determine the spum and the integral spum of complete symmetric bipartite graphs. Hartsfield and Smyth [7] showed that the sum number $\sigma(\mathcal{K}_{n,n}) = 2n-1$ for $n \geq 2$. Yan and Liu [12] showed that the integral sum number $\zeta(\mathcal{K}_{n,n}) = 2n-1$ for $n \geq 2$. We show that, for $n \geq 2$,

$$\texttt{spum}\ \mathcal{K}_{n,n} = 7n-7, \qquad \texttt{integral spum}\ \mathcal{K}_{n,n} = 7n-7.$$

Theorem 5.1. *For $n \geq 2$,*

$$\texttt{spum}\ \mathcal{K}_{n,n} \leq 7n-7, \qquad \texttt{integral spum}\ \mathcal{K}_{n,n} \leq 7n-7.$$

Proof. Let G be the graph induced by the set of labels $L = [3(n-1), 4(n-1)] \cup [5(n-1), 6(n-1)] \cup [8(n-1), 10(n-1)]$. We show that the two sets of n vertices with labels in $I_1 = [3(n-1), 4(n-1)]$ and in $I_2 = [5(n-1), 6(n-1)]$ form the two bipartite sets while the $2n-1$ vertices with labels in $I_3 = [8(n-1), 10(n-1)]$ are isolated. Let $S_1 + S_2 = \{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2, x_1 \neq x_2\}$. Then $I_1 + I_1 = [6n-5, 8n-9] \cap L = \emptyset$, $I_1 + I_2 = [8(n-1), 10(n-1)] = I_3$, and $I_2 + I_2 = [10n - 9, 12n - 13] \cap L = \emptyset$, proving that the graph vertices with labels in $I_1 \cup I_2$ are isomorphic to $\mathcal{K}_{n,n}$. Moreover $(L + a) \cap L = \emptyset$ for $a \in I_3$, implying that the vertices with labels in I_3 are isolated.

Since $\zeta(\mathcal{K}_{n,n}) = \sigma(\mathcal{K}_{n,n}) = 2n - 1$, $\texttt{integral spum}\, \mathcal{K}_{n,n} \le \texttt{spum}\, \mathcal{K}_{n,n} \le 10(n-1) - 3(n-1) = 7n - 7$. ■

Theorem 5.2. *For $n \ge 2$,*

$$\texttt{spum}\, \mathcal{K}_{n,n} \ge 7n - 7, \qquad \texttt{integral spum}\, \mathcal{K}_{n,n} \ge 7n - 7.$$

Proof. Since $\sigma(\mathcal{K}_{n,n}) = \zeta(\mathcal{K}_{n,n}) = 2n-1$ and $\texttt{integral spum}$ of a graph is always bounded above by its $\texttt{spum}$, it suffices to prove that $\texttt{integral spum}\, \mathcal{K}_{n,n} \ge 7n - 7$.

Let G be an integral sum graph consisting of $\mathcal{K}_{n,n}$ together with $2n - 1$ isolated vertices. Let L be a labelling of G for which $\max L - \min L = \texttt{integral spum}\, \mathcal{K}_{n,n}$. Let the labels within L corresponding to the bipartition of $\mathcal{K}_{n,n}$ be the sets $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, each written in increasing order. By multiplying by -1 if necessary, we can ensure that the number among a_1, a_n, b_1, b_n with largest absolute value is positive. We relabel the numbers so that the largest label is b_n. This ensures $b_n > a_n$ and $b_n \ge |a_1|$.

Yan and Liu [12, Theorem 3.1] showed that the $2n-1$ isolated vertices of G have labels that we can partition into sets C and D as follows:

$$C = \{a_1 + b_i : 1 \le i \le n\} = B + a_1, \quad D = \{b_n + a_i : 2 \le i \le n\} = \big(A \setminus \{a_1\}\big) + b_n.$$

Observe that the elements in each of the sets C, D are in increasing order, and that $\max C < \min D$. Hence the sets A, B, C, D partition L.

Interchanging the roles of a_i's and b_i's yield the sets

$$E = \{b_1 + a_i : 1 \le i \le n\} = A + b_1, \quad F = \{a_n + b_i : 2 \le i \le n\} = \big(B \setminus \{b_1\}\big) + a_n.$$

We again see that the elements in each of the sets E, F are in increasing order, and that $\max E < \min F$. Hence we must have

$$a_1 + b_i = b_1 + a_i \;\; (1 \le i \le n), \quad b_n + a_i = a_n + b_i \;\; (2 \le i \le n). \tag{3}$$

Thus $b_i - a_i$ is constant for $1 \le i \le n$; write $b_i - a_i = d > 0$.

We claim that the vertex labelled $a_i + b_j$ is isolated for all choices of $i, j \in \{1, \dots, n\}$. Thus we must show that $a_i + b_j \notin A \cup B$ for all choices of $i, j \in \{1, \dots, n\}$. This is true for $i = 1$ since $a_1 + b_j \in C$ for $j \in \{1, \dots, n\}$. If $i > 1$ and $a_i + b_j \in A \cup B$, then the isolated vertex labelled $a_i + b_n$ is adjacent to the vertex labelled b_j, which is impossible. Hence the claim.

We recall that L is the disjoint union of the sets A, B, C, D, and that $\big|A\big| = \big|B\big| = \big|C\big| = n$ and $\big|D\big| = n - 1$. Hence $\big|L\big| = 4n - 1$. Let $S_1 = D - |a_1|$, $S_2 = D - |b_1|$ and $S_3 = (\operatorname{sgn} a_1)\big(A \setminus \{a_n\}\big) + a_n$. Observe that $\big|S_1\big| = \big|S_2\big| = \big|S_3\big| = n - 1$.

We show that each of the sets S_1, S_2, S_3 is contained in the interval $[\min L, \max L]$.

It is clear that $\max S_i \le \max D \le \max L$ and that $\min S_i \ge \min D - \max\{|a_1|, |b_1|\} = b_n + a_2 - \max\{|a_1|, |b_1|\}$ for $i = 1, 2$. If $a_1 > 0$, then $\min D - \max\{|a_1|, |b_1|\} = b_n + a_2 - b_1 > a_2 > a_1 = \min L$. If $a_1 < 0 < b_1$, then $\min D - \max\{|a_1|, |b_1|\} = b_n + a_2 - \max\{-a_1, b_1\} \ge a_2 > a_1 = \min L$ since $b_n \ge -a_1$. If $b_1 < 0$, then $\min D - \max\{|a_1|, |b_1|\} = b_n + a_2 + a_1 > (b_n + a_1) + a_1 \ge a_1 > a_1 + b_1 = \min L$. Thus S_1 and S_2 are both contained in the interval $[\min L, \max L]$.

If $a_1 > 0$, then $\min L = a_1 < a_2 + a_n = \min S_3 < \max S_3 = a_{n-1} + a_n < b_n + a_n = \max L$. If $a_1 < 0$, then $\min L = \min\{a_1, a_1 + b_1\} < 0 < a_n - a_{n-1} = \min S_3 < \max S_3 = a_n - a_2 < a_n - a_1 \le a_n + b_n \le \max L$. Hence S_3 is contained in the interval $[\min L, \max L]$.

For any sets S_1, S_2, S_3, S_4, we have by Principle of Inclusion & Exclusion

$$\left|\bigcup_{i=1}^{4} S_i\right| \ge \sum_{i=1}^{4} \big|S_i\big| - \sum_{1 \le i < j \le 4} \big|S_i \cap S_j\big|.$$

Applying this inequality to the sets considered above and taking $S_4 = L$, we get $\big|S_1 \cup S_2 \cup S_3 \cup L\big| \ge 3(n-1) + (4n-1) - \Sigma$, where Σ denotes $\sum_{1 \le i < j \le 4} \big|S_i \cap S_j\big|$. Therefore $\Sigma \le 3$ would imply that at least $7n - 7$ integers lie within the interval $[\min L, \max L]$, proving our claim.

We consider the sizes of the six sets $S_i \cap S_j$ with $1 \le i < j \le 4$.

Case (i) $(S_1 \cap S_2)$

Suppose $a_1 > 0$ or $b_1 < 0$. If $x \in S_1 \cap S_2$, then $b_n + a_i \pm a_1 = x = b_n + a_j \pm b_1$ for some $i, j \ne 1$. Thus $a_i = a_j \pm d$, so that $b_i = a_j$ or $a_i = b_j$, both of which are impossible.

Suppose $a_1 < 0 < b_1$. If $x \in S_1 \cap S_2$, then $b_n + a_i + a_1 = b_n + a_j - b_1$ for some $i, j \ne 1$. This is impossible because then the isolated vertex labelled $a_i + b_1$ is adjacent to the vertex labelled a_1. Hence $S_1 \cap S_2 = \emptyset$ in all cases.

Case (ii) $(S_1 \cap S_3)$

Suppose $a_1 > 0$. If $x \in S_1 \cap S_3$, then $b_n + a_i - a_1 = x = a_j + a_n$ with $2 \le i \le n$ and $1 \le j \le n - 1$. Hence $a_1 + a_j = b_i$, which can possibly hold only if $j = 1$ since the vertices with labels a_1 and a_j belong to the same partite set. Hence $\big|S_1 \cap S_3\big| \le 1$.

Suppose $a_1 < 0$. If $x \in S_1 \cap S_3$, then $b_n + a_i + a_1 = a_n - a_j$ with $2 \le i \le n$ and $1 \le j \le n-1$. Hence $(a_1 + b_i) + (a_j + b_k) = b_k$ holds for $k \in \{1, \ldots, n\}$. Now the isolated vertex with label $a_1 + b_i$ is adjacent to the isolated vertex with label $a_j + b_k$ for $k \in \{1, \ldots, n\}$, and this is impossible. Hence $S_1 \cap S_3 = \emptyset$.

Case (iii) $(S_2 \cap S_3)$

Suppose $a_1 > 0$. If $x \in S_2 \cap S_3$, then $b_n + a_i - b_1 = a_j + a_n$ with $2 \le i \le n$ and $1 \le j \le n-1$. Hence $a_j + b_1 = b_i$. But this is impossible since the isolated vertex labelled $a_j + b_1$ has the same label as the non-isolated vertex b_i.

Suppose $a_1 < 0 < b_1$. If $x \in S_2 \cap S_3$, then $b_n + a_i - b_1 = a_n - a_j$ with $2 \le i \le n$ and $1 \le j \le n-1$. But this is impossible since the isolated vertex labelled $a_j + b_i$ has the same label as the non-isolated vertex b_1.

Suppose $b_1 < 0$. If $x \in S_2 \cap S_3$, then $b_n + a_i + b_1 = a_n - a_j$ with $2 \le i \le n$ and $1 \le j \le n-1$. Hence $(a_j + b_1) + (a_k + b_i) = a_k$ holds for $k \in \{1, \ldots, n\}$. Now the isolated vertex with label $a_j + b_1$ is adjacent to the isolated vertex with label $a_k + b_i$ for $k \in \{1, \ldots, n\}$, and this is impossible. Hence $S_2 \cap S_3 = \emptyset$ in all cases.

Case (iv) $(S_1 \cap L)$

Suppose $a_1 > 0$. Observe that $S_1 \cap (A \cup B) = \emptyset$ since $\max A < \max B < \min S_1$. If $x \in S_1 \cap (C \cup D)$, then the vertex with label a_1 is adjacent to a vertex with label in $C \cup D$, which is impossible since vertices with labels in $C \cup D$ are isolated. Hence $S_1 \cap (C \cup D) = \emptyset$.

Suppose $a_1 < 0$. If $S_1 \cap L \neq \emptyset$, then $b_n + a_i + a_1 \in L$ for some $i \neq 1$. This is impossible since then the isolated vertex labelled $b_n + a_i$ is adjacent to the vertex labelled a_1. Hence $S_1 \cap L = \emptyset$ in all cases.

Case (v) $(S_2 \cap L)$

Suppose $b_1 > 0$. Observe that $S_2 \cap A = \emptyset$ since $\max A < \min S_2$. If $S_2 \cap (C \cup D) \neq \emptyset$, then the vertex with label b_1 is adjacent to a vertex with label in $C \cup D$. This is impossible since vertices with labels in $C \cup D$ are isolated. Hence $S_2 \cap (C \cup D) = \emptyset$. If $S_2 \cap B \neq \emptyset$, then $b_1 + b_j \in D$ for some $j \in \{1, \ldots, n\}$. But then the vertices with labels b_1 and b_j are adjacent, and this is impossible since these vertices are from the same partite set unless $j = 1$. Thus S_2 and B can have at most the vertex with label b_1 in common. Hence $\left|S_2 \cap L\right| \le 1$ in this case.

Suppose $b_1 < 0$. If $S_2 \cap L \neq \emptyset$, then $b_n + a_i + b_1 \in L$ for some $i \neq 1$. But then the isolated vertex labelled $b_n + a_i$ is adjacent to the vertex labelled b_1, which is impossible. Hence $S_2 \cap L = \emptyset$ in this case.

Case (vi) $(S_3 \cap L)$

Suppose $a_1 > 0$. If $S_3 \cap L \neq \emptyset$, then $a_i + a_n \in L$ for some $i \neq n$. But then the vertices with labels a_i and a_n are adjacent, which is impossible since they belong to the same partite set. Hence $S_3 \cap L = \emptyset$ in this case.

Suppose $a_1 < 0$. If $S_3 \cap (A \cup C \cup D) \neq \emptyset$, then $a_n - a_i \in A \cup C \cup D$ for some $i \neq n$. But then the vertex with label a_i is adjacent to some vertex with label in $A \cup C \cup D$. This is impossible as neighbours of vertex with label a_i must have labels in B, with a possible exception in case $a_n = 2a_i$. If $S_3 \cap B \neq \emptyset$, then $a_n - a_i = b_j$ for some $i \neq n$ and $j \in \{1, \ldots, n\}$. This is impossible since $a_i + b_j$ is the label of an isolated vertex whereas a_n is the label of a non-isolated vertex. Thus $\left|S_3 \cap L\right| \le 1$ in this case.

Since only three cases (ii), (v), and (vi) above lead to $\left|S_i \cap S_j\right| \le 1$, it follows that $\Sigma \le 3$, as desired. This completes the proof. ■

6. Cycles

Harary [5] showed that the sum number $\sigma(\mathcal{C}_n) = 2$, except that $\sigma(\mathcal{C}_4) = 3$. Sharary [9] showed that the integral sum number $\zeta(\mathcal{C}_n) = 0$ for $n \neq 4$. We show that, for $n \ge 4$,

$$2n - 2 \le \operatorname{spum} \mathcal{C}_n \le 2n - 1,$$

and for $n \ge 13$,

$$\operatorname{spum} \mathcal{C}_n = 2n - 1.$$

Theorem 6.1. *For $n \ge 4$, $\operatorname{spum} \mathcal{C}_n \le 2n - 1$.*

Proof. For odd $n \ge 5$, let G_1 be the graph induced by the set of labels $L_1 = \mathcal{L}(G_1) = [n-3, 2n-4] \cup \{3n-6, 3n-4\}$. We claim that the graph induced by $[n-3, 2n-4]$ is isomorphic to $\mathcal{C}_n$ by showing that the vertices with labels in the sequence

$$n-3, n-1, 2n-5, n+1, 2n-7, n+3, 2n-9, \ldots, 2n-4, n-2, n-3 \tag{4}$$

form a cycle, with the vertices labelled $3n-6$ and $3n-4$ isolated. We first show that the sequence in Eq. (4) with the first and last terms removed form a path with $n-1$ vertices. We note that this sequence alternates between even and odd integers in the interval $I_1 = [n-2, 2n-4]$, with the even integers starting at $n-1$ and increasing and the odd integers starting at $2n-5$ and decreasing. It is easy to see that consecutive sums of labels alternately yield $3n-6$ and $3n-4$, thereby forming a path. Now the sum of two distinct integers both taken from the interval I_1 lies in the interval $[2n-3, 4n-9]$ and $[2n-3, 4n-9] \cap L_1 = \{3n-6, 3n-4\}$. Hence, for $a \in I_1 \setminus \{n-2, n-1\}$, there exist $b = (3n-4) - a \in I_1$ with $c = (3n-6) - a \in I_1$. Since the vertices with labels in I_1 form a path, there are no other edges between these vertices. Thus the graph induced by the vertices with labels in I_1 is isomorphic to $\mathcal{P}_{n-1}$ with endpoints labelled $n-1$ and $n-2$. The vertex with label $n-3$ is adjacent to both endpoints of the path (because $2n-5, 2n-4 \in L_1$), and to no other vertex (because $[n, 2n-4] + (n-3) \cap L_1 = \emptyset$). Moreover it is easy to see that the vertices with the labels $3n-6$ and $3n-4$ are isolated. This completes the proof of claim that $G_1 \cong \mathcal{C}_n$.

Hence $\operatorname{spum} \mathcal{C}_n \leq \max L_1 - \min L_1 = (3n-4) - (n-3) = 2n-1$ when n is odd.

For even $n \geq 4$, let G_2 be the graph induced by the set of labels $L_2 = \mathcal{L}(G_2) = [n-2, 2n-3] \cup \{3n-5, 3n-3\}$. We claim that the graph induced by $[n-2, 2n-3]$ is isomorphic to $\mathcal{C}_n$ by showing that the vertices with labels in the sequence

$$n-2, 2n-3, n, 2n-5, n+2, 2n-7, \ldots, 2n-4, n-1, n-2 \tag{5}$$

form a cycle, with the vertices labelled $3n-5$ and $3n-3$ isolated. We note that this sequence alternates between even and odd integers in the interval $I_2 = [n-2, 2n-3]$, with the even integers starting at $n-2$ and increasing and the odd integers starting at $2n-3$ and decreasing. It is easy to see that consecutive sums of labels alternately yield $3n-5$ and $3n-3$, except for the last sum which is $2n-3$, thereby forming a cycle. Now the sum of two distinct integers both taken from the interval I_2 lies in the interval $[2n-3, 4n-7]$ and $[2n-3, 4n-7] \cap L_2 = \{2n-3, 3n-5, 3n-3\}$. Hence, for $a \in I_2$, there exist $b = (3n-5) - a \in I_2$ with $c = (3n-3) - a \in I_2$ for $a \neq n-2, n-1$ and $c = (2n-3) - a$ for $a = n-2, n-1$. Since the vertices with labels in I_2 form a cycle given by Eq. (5), there are no other edges between these vertices. Moreover it is easy to see that the vertices with the labels $3n-5$ and $3n-3$ are isolated. This completes the proof of claim that $G_2 \cong \mathcal{C}_n$.

Hence $\operatorname{spum} \mathcal{C}_n \leq \max L_2 - \min L_2 = (3n-3) - (n-2) = 2n-1$ when n is even. ■

Remark 6.1. Theorems 2.1 and 6.1 imply $2n-2 \leq \operatorname{spum} \mathcal{C}_n \leq 2n-1$ for $n \geq 4$.

Theorem 6.2. *For $n \geq 13$, $\operatorname{spum} \mathcal{C}_n \geq 2n-1$.*

Proof. Let G be a sum graph consisting of $\mathcal{C}_n$ together with two isolated vertices. Let $v_1, \ldots, v_n$ be the vertices on $\mathcal{C}_n$ and let x, y be the isolated vertices. Let L be a labelling of G for which $\max L - \min L = \operatorname{spum} \mathcal{C}_n$. Let $S = \{a_1, \ldots, a_n\}$, written in increasing order, denote the labellings within L of vertices corresponding to $\mathcal{C}_n$. Let $\ell(v_i) = a_i$ for $1 \leq i \leq n$, and let $\ell(x) = a$, $\ell(y) = b$, with $a < b$. Let

$$S_1 = S \cap [a_1, 2a_1], \quad S_2 = S \cap [2a_1 + 1, a_n], \quad S_3 = S_2 - a_1, \quad T = [a_1, a_n] \setminus S. \tag{6}$$

From Theorem 2.1 we know that $\operatorname{spum} \mathcal{C}_n \geq 2n-2$. We show that $\operatorname{spum} \mathcal{C}_n = 2n-2$ is not possible for $n \geq 13$. We first show that $\operatorname{spum} \mathcal{C}_n = 2n-2$ implies $[a_1, 2a_1] \subset S$, and then use this to show that both $a_1 \leq \frac{5n+2}{12}$ and $a_1 \geq n-7$ must hold. The lower and upper bounds for a_1 can simultaneously hold only when $n-7 \leq \frac{5n+2}{12}$, or when $n \leq 12$.

We first show that $\operatorname{spum} \mathcal{C}_n = 2n-2$ implies $[a_1, 2a_1] \subset S$. If this was not the case, Eq. (1) in this special case would be replaced by

$$n - a_1 \leq |S_3| = |S_3 \cap [a_1, a_n]| = |S_3 \cap S| + |S_3 \cap T|.$$

This is the same as Eq. (1) except that the lower bound for $|S_3|$ has been replaced by $n - a_1$. Therefore, the arguments in the two paragraphs immediately following Eq. (1) now imply $\max L - \min L \geq 2n-1$ since $\Delta = \delta$. This contradicts our assumption that $\operatorname{spum} \mathcal{C}_n = 2n-2$. Hence $[a_1, 2a_1] \subset S$.

Now assuming $[a_1, 2a_1] \subset S$ and $\operatorname{spum} \mathcal{C}_n = 2n-2$, we prove the following two claims that give upper and lower bounds on a_1. Together they imply a feasible value of a_1 exists only for $n \leq 16$, which finishes the proof of this theorem.

Claim 1. $\operatorname{spum} \mathcal{C}_n = 2n-2$ *implies* $a_1 \leq \frac{5n+2}{12}$.

Recall that under the assumption $\operatorname{spum} \mathcal{C}_n = 2n-2$, $[a_1, 2a_1] \subset S$, and so $\ell(v_{a_1+1}) = 2a_1$. Since v_{a_1+1} has a neighbour with label greater than a_1, there is a vertex with label greater than $3a_1$. Hence $a_n > 3a_1$, and $2n-2 = \operatorname{spum} \mathcal{C}_n \geq (3a_1 + 1 + a_1 + 1) - a_1 = 3a_1 + 2$. Therefore $a_1 \leq \frac{2n-4}{3}$.

We claim that $|S \cap [2a_1 + 1, 3a_1]| \leq 2$. Suppose, to the contrary, that $|S \cap [2a_1 + 1, 3a_1]| \geq 3$, and that a_i, a_j, a_k are the labels of three of these vertices in $[2a_1 + 1, 3a_1]$. Then vertex v_1 is adjacent to each of the three vertices with labels $a_i - a_1, a_j - a_1, a_k - a_1$, contradicting the assumption that $d(v_1) = 2$. This proves our claim.

We claim that $|S \cap [3a_1 + 1, 4a_1]| \leq 2$. Again suppose, to the contrary, that $|S \cap [3a_1 + 1, 4a_1]| \geq 3$, and that b_i, b_j, b_k are the labels of three of these vertices in $[3a_1 + 1, 4a_1]$. Then vertex v_{a_1+1} with label $2a_1$ is adjacent to each of the three vertices with labels $b_i - 2a_1, b_j - 2a_1, b_k - 2a_1$, contradicting the assumption that $d(v_{a_1+1}) = 2$. This proves our claim.

Since $[a_1, 2a_1] \subset S$ and S has at most two elements in common with each of the intervals $[2a_1 + 1, 3a_1]$ and $[3a_1 + 1, 4a_1]$, $|S \cap [a_1, 4a_1]| \leq a_1 + 5 \leq \frac{2n+11}{3}$. Thus there are at least $n - \frac{2n+11}{3} = \frac{n-11}{3}$ elements of S in the interval $[4a_1 + 1, a_n]$. Hence $a_n \geq 4a_1 + \frac{n-11}{3}$, and $\operatorname{spum} \mathcal{C}_n \geq (4a_1 + \frac{n-11}{3} + a_1 + 1) - a_1 = 4a_1 + \frac{n-8}{3}$.

We conclude that $4a_1 + \frac{n-8}{3} \leq 2n-2$, so that $a_1 \leq \frac{5n+2}{12}$, as claimed.

Claim 2. $\mathtt{spum}\ \mathcal{C}_n = 2n-2$ *implies* $a_1 \geq n-7$.

We recall the sets defined at the beginning of this proof in Eq. (6). We further define sets

$$S_2' = S \cap [2a_1+1, 3a_1], \quad S_2'' = S \cap [3a_1+1, a_n], \quad S_3' = S_2' - a_1, \quad S_3'' = S_2'' - a_1, \quad S_4'' = S_2'' - 2a_1.$$

Note that S_2 is the disjoint union of S_2' and S_2'' and S_3 is the disjoint union of S_3' and S_3''. Recall that under the assumption $\mathtt{spum}\ \mathcal{C}_n = 2n-2$, $[a_1, 2a_1] \subset S$, and so $\ell\left(v_{a_1+1}\right) = 2a_1$.

Suppose $s \in S \cap S_3$. Then $s + a_1 \in S_2 \subset S$ with $s \in S$, so that each vertex with label in $S \cap S_3$ is a neighbour of v_1. Again if $t \in T \cap S_3 \cap S_4''$, then $t + a_1 \in S_2 \subset S$ and $t + 2a_1 \in S_2'' \subset S$. So the vertex labelled $t + a_1$ is a neighbour of v_1.

Suppose $s \in S \cap S_4''$. Then $s + 2a_1 \in S_2'' \subset S$ with $s \in S$, and so v_{a_1+1} has at least as many as $\left|S \cap S_4''\right|$ neighbours.

Since $d(v_1) = d\left(v_{a_1+1}\right) = 2$, we have

$$\left|S \cap S_3\right| \leq 2, \quad \left|S \cap S_4''\right| \leq 2, \quad \left|S_3 \cap S_4''\right| \leq 2. \tag{7}$$

Since S, T partition the interval $[a_1, a_n]$ and S_3, S_4'' lie within $[a_1, a_n]$, we have

$$\left|S \cap S_3\right| + \left|T \cap S_3\right| = \left|S_3\right| = \left|S_2\right| = \left|S\right| - \left|[a_1, 2a_1]\right| = n - (a_1+1) \tag{8}$$

and

$$\left|S \cap S_4''\right| + \left|T \cap S_4''\right| = \left|S_4''\right| = \left|S_2''\right| = \left|S\right| - \left|S \cap [a_1, 3a_1]\right| \geq n - (a_1+1) - 2, \tag{9}$$

the last inequality because $\left|S \cap [2a_1+1, 3a_1]\right| \leq 2$, as shown earlier.

Eqs. (7), (8), and (9) give

$$\left|T \cap S_3\right| \geq n - a_1 - 3, \quad \left|T \cap S_4''\right| \geq n - a_1 - 5. \tag{10}$$

Using Eqs. (7) and (10), and the fact that $\left|T\right| = (a_n - a_1 + 1) - n$ in

$$\left|T\right| \geq \left|T \cap (S_3 \cup S_4'')\right| = \left|T \cap S_3\right| + \left|T \cap S_4''\right| - \left|T \cap S_3 \cap S_4''\right| \geq \left|T \cap S_3\right| + \left|T \cap S_4''\right| - \left|S_3 \cap S_4''\right|,$$

we have

$$a_n \geq 3n - a_1 - 11. \tag{11}$$

To complete the proof of this Claim 2, suppose $v_1 \leftrightarrow v_n$. Since $a_n \in S_3 \cup S_4''$, Eq. (7) may be replaced by

$$\left|S \cap S_3\right| \leq 1, \quad \left|S \cap S_4''\right| \leq 1, \quad \left|S_3 \cap S_4''\right| \leq 1.$$

Consequently Eqs. (8) and (9) now imply that Eq. (10) may be replaced by

$$\left|T \cap S_3\right| \geq n - a_1 - 2, \quad \left|T \cap S_4''\right| \geq n - a_1 - 4,$$

and Eq. (11) by

$$a_n \geq 3n - a_1 - 8.$$

Hence $\mathtt{spum}\ \mathcal{C}_n \geq (3n - a_1 - 8) + (a_1 + 1) - a_1 = 3n - a_1 - 7$.

If $v_1 \not\leftrightarrow v_n$, from Eq. (11) we get $\mathtt{spum}\ \mathcal{C}_n \geq (3n - a_1 - 11) + (a_1 + 2) - a_1 = 3n - a_1 - 9$. In any case, we have $2n - 2 \geq 3n - a_1 - 9$, so that $a_1 \geq n - 7$, as claimed.

This completes the proof of the theorem. ■

Melnikov & Pyatkin [8, Lemma 2, pp. 240–242] provided an integral labellings for cycles $\mathcal{C}_n$ with $n \geq 5$. For each $k \geq 1$, they showed that the set

$$[-17k, -16k+1] \cup [-12k, -12k+1] \cup \{-5k, -k-1\} \cup [4k, 5k] \cup \{16k-1, 17k\}$$

induces the cycle $\mathcal{C}_{2k+9}$, and the set

$$[-5k-8, -4k-7] \cup [-3k-6, -3k-5] \cup [-k-2, -k-1] \cup [k+2, 2k+2] \cup \{4k+7\}$$

induces the cycle $\mathcal{C}_{2k+8}$. They also provided integral labels for $\mathcal{C}_n$ with $n \in \{5, 6, 7, 8, 9\}$. Thus we have the following upper bound for $\mathtt{integral\ spum}\ \mathcal{C}_n$ for $n \geq 10$.

Theorem 6.3 (*Melnikov & Pyatkin, [8]*). *For* $n \geq 10$,

$$\mathtt{integral\ spum}\ \mathcal{C}_n \leq \begin{cases} 17(n-9) & \text{if } n \text{ is odd;} \\ \frac{3}{2}(3n-14) & \text{if } n \text{ is even.} \end{cases}$$

Table 2
Table of results for `integral spum` of cycles.

n	integral spum labelling	integral spum $\mathcal{C}_n$
5	−**3**, **2**, −1, −2, 1	5
6	−**5**, **3**, −2, −3, 1, 2	8
7	−**7**, **4**, −3, −4, 1, −5, 2	11
8	−**11**, **3**, −10, −1, −7, −3, −8, 1	14
9	−**8**, **9**, −1, −5, −3, 8, 1, −6, 5	17
10	−**13**, **4**, −2, −10, −3, −9, −4, 2, −12, 3	17
11	−**5**, **16**, −4, 15, −3, −2, 5, 11, 4, 12, 3	21
12	−**6**, **19**, −5, 18, −4, 17, −3, 6, 13, 4, 14, 3	25
13	−**6**, **20**, −5, 19, −4, 7, −3, 6, 14, 5, 15, 4, 3	26

Theorem 6.4. *For $n \geq 3$, $\texttt{integral spum}\ \mathcal{C}_n \geq 2n-5$.*

Proof. This follows directly from Theorem 2.2. ■

We are unable to determine a better upper bound for $\texttt{integral spum}\ \mathcal{C}_n$ than the one in Theorem 6.3, but we make the following conjecture based on the limited evidence of Table 2.

Conjecture 6.1. *For $n \geq 9$,*

$$\texttt{integral spum}\ \mathcal{C}_n = \begin{cases} \frac{5}{2}(n-3)+1 & \textit{if } n \textit{ is odd;} \\ 2n-2 & \textit{if } n \textit{ is even.} \end{cases}$$

7. Paths

Harary showed that the sum number $\sigma(\mathcal{P}_n) = 1$ in [5] and that the integral sum number $\zeta(\mathcal{P}_n) = 0$ in [6]. For $n \geq 9$, we show that

$$2n-3 \leq \texttt{spum}\ \mathcal{P}_n \leq \begin{cases} 2n+1 & \text{if } n \text{ is odd;} \\ 2n+2 & \text{if } n \text{ is even,} \end{cases}$$

and for $n \geq 7$, we show that

$$2n-5 \leq \texttt{integral spum}\ \mathcal{P}_n \leq \begin{cases} \frac{5}{2}(n-3) & \text{if } n \text{ is odd;} \\ 2n-3 & \text{if } n \text{ is even.} \end{cases}$$

Theorem 7.1. *For $n \geq 9$,*

$$\texttt{spum}\ \mathcal{P}_n \leq \begin{cases} 2n+1 & \textit{if } n \textit{ is odd;} \\ 2n+2 & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. For odd $n \geq 9$, let G_1 be the graph induced by the set of labels $L_1 = \mathcal{L}(G_1) = \{1, 3, 5, \ldots, 2n-7\} \cup \{2n-6, 2n-3, 2n+1, 2n+2\}$. We claim that the graph induced by $L_1 \setminus \{2n+2\}$ is isomorphic to $\mathcal{P}_n$ by showing that the vertices with labels in the sequence

$$2n+1, 1, 2n-7, 9, \ldots, 11, 2n-9, 3, 2n-6, 7, 2n-13, 15, 2n-21, \ldots, 2n-11, 5, 2n-3 \tag{12}$$

form a path when n is of the form $4k+1$, and by showing that the vertices with labels in the sequence

$$2n+1, 1, 2n-7, 9, \ldots, 15, 2n-9, 7, 2n-6, 3, 2n-13, 11, 2n-21, \ldots, 2n-11, 5, 2n-3 \tag{13}$$

form a path when n is of the form $4k+3$. In both cases, the vertex labelled $2n+2$ is isolated.

For the case $n = 4k+1$, the subsequence of odd subscripts consists of two arithmetic progressions each with common difference 8. One starts with $\ell_1 = 2n+1$ and ends with 3 (there are $\frac{n+3}{4}$ terms), the other starts with $\ell_n = 2n-3$ and ends with 7 (there are $\frac{n-1}{4}$ terms). The subsequence of even subscripts also consists of two arithmetic progressions each with common difference 8, but with a term in the middle: $\ell_{(n+3)/2} = 2n-6$. One of these progressions starts with $\ell_2 = 1$ and ends with $2n-9$ (there are $\frac{n-1}{4}$ terms), the other starts with $\ell_{n-1} = 5$ and ends with $2n-13$ (there are $\frac{n-5}{4}$ terms). We can define the sequence in Eq. (12) by:

$$\ell_{2i+1}=\begin{cases}(2n+1)-8i & \text{if } 0\le i\le \frac{n-1}{4},\\ -(2n-1)+8i & \text{if } \frac{n+3}{4}\le i\le \frac{n-1}{2},\end{cases}\qquad \ell_{2i}=\begin{cases}8i-7 & \text{if } 1\le i\le \frac{n-1}{4},\\ 2n-6 & \text{if } i=\frac{n+3}{4},\\ (4n+1)-8i & \text{if } \frac{n+7}{4}\le i\le \frac{n-1}{2}.\end{cases}$$

Observe that $\ell_i+\ell_{i+1}$ is alternately $2n+2$ and $2n-6$, except that $\ell_{(n-1)/4}+\ell_{(n+3)/4}=2n-3$ and $\ell_{(n+3)/4}+\ell_{(n+7)/4}=2n+1$. Thus the vertices with labels given in Eq. (12) form a path.

We now prove that $v_i \not\leftrightarrow v_j$ when $|i-j|>1$. Observe that exactly two of the labels are even (in fact, multiples of 4); the label $2n-6$ corresponds to $v_{(n+3)/4}$ and the label $2n+2$ corresponds to an isolated vertex. Observe also that $\ell_{\text{odd}}\equiv 3 \pmod 4$ and $\ell_{\text{even}}\equiv 1 \pmod 4$, for even subscript $\neq \frac{n+3}{2}$. Hence, by considering residue classes modulo 4, we see that $v_{2i-1}\not\leftrightarrow v_{2j-1}$ for $i\neq j$ and $v_{2i}\not\leftrightarrow v_{2j}$ for $i\neq j$, $i,j\neq \frac{n+3}{4}$. Thus $v_{2i-1}\leftrightarrow v_{2j}$ if and only if $\ell_{2i-1}+\ell_{2j}\in\{2n-6,2n+2\}$, so that each of the vertices (possibly except $v_{(n+3)/4}$) must have degree 0, 1, or 2. Since these vertices lie on the path with labels given by Eq. (14), we need to show that $d(v_1)=d(v_n)=1$, $d(v_{(n+3)/2})=2$; the vertex with label $2n+2=\max L_1$ is isolated. We note that in order that v_i is a neighbour of v_1 (respectively, $v_{(n+3)/2}$, v_n), ℓ_i must belong to $\{1\}$ (respectively, $\{3,7,8\}$, $\{4,5\}$). The proof of the claim that the graph induced by $L_1\setminus\{2n+2\}$ is isomorphic to $\mathcal{P}_n$ is complete with the observation that $1,3,5,7\in L_1$ and $4\notin L_1$ for the case $n=4k+1$.

The case $n=4k+3$ is almost identical to the case $n=4k+1$ discussed above. The subsequence of odd subscripts consists of two arithmetic progressions each with common difference 8. One starts with $\ell_1=2n+1$ and ends with 7 (there are $\frac{n+1}{4}$ terms), the other starts with $\ell_n=2n-3$ and ends with 3 (there are $\frac{n+1}{4}$ terms). The subsequence of even subscripts also consists of two arithmetic progressions each with common difference 8, but with a term in the middle: $\ell_{(n+1)/2}=2n-6$. One of these progressions starts with $\ell_2=1$ and ends with $2n-9$ (there are $\frac{n-1}{4}$ terms), the other starts with $\ell_{n-1}=5$ and ends with $2n-13$ (there are $\frac{n-5}{4}$ terms). We can define the sequence in Eq. (13) by:

$$\ell_{2i+1}=\begin{cases}(2n+1)-8i & \text{if } 0\le i\le \frac{n-3}{4},\\ -(2n-1)+8i & \text{if } \frac{n+1}{4}\le i\le \frac{n-1}{2},\end{cases}\qquad \ell_{2i}=\begin{cases}8i-7 & \text{if } 1\le i\le \frac{n-3}{4},\\ 2n-6 & \text{if } i=\frac{n+1}{4},\\ (4n+1)-8i & \text{if } \frac{n+5}{4}\le i\le \frac{n-1}{2}.\end{cases}$$

The details of the proof for this case are identical to the case when $n=4k+1$ and is omitted. The proof of the claim that the graph induced $L_1\setminus\{2n+2\}$ is isomorphic to $\mathcal{P}_n$ is complete for the case when n is odd.

Hence $\operatorname{spum}\mathcal{P}_n\le \max L_2-\min L_2=(2n+2)-1=2n+1$ when n is odd.

For even $n\ge 10$, let G_2 be the graph induced by the set of labels $L_2=\mathcal{L}(G_2)=\{1,3,5,\dots,2n-9\}\cup\{2n-5,2n-4,2n-1,2n,2n+3\}$. We claim that the graph induced by $L_2\setminus\{2n+3\}$ is isomorphic to $\mathcal{P}_n$ by showing that the vertices with labels in the sequence

$$2n-1,1,2n-5,5,2n-9,9,\dots,2n-4,3,2n \tag{14}$$

form a path, with the vertex labelled $2n+3$ isolated. We note that the labels are given by $\ell_{2i-1}=(2n-1)-4(i-1)=2n+3-4i$ for $i\in\{1,\dots,\frac{n}{2}\}$ and $\ell_{2i}=1+4(i-1)=4i-3$ for $i\in\{1,\dots,\frac{n}{2}-2\}$, with $\ell_{n-2}=2n-4$ and $\ell_n=2n$. Observe that $\ell_i+\ell_{i+1}$ is alternately $2n$ and $2n-4$, except that the last two sums are $2n-1$ and $2n+3$. Thus the vertices with labels given in Eq. (14) form a path.

We now prove that $v_i\not\leftrightarrow v_j$ when $|i-j|>1$. Observe that exactly two of the labels are even (in fact, multiples of 4), and these correspond to the vertices v_{n-2} and v_n. Observe also that $\ell_{\text{odd}}\equiv 3\pmod 4$ and $\ell_{\text{even}}\equiv 1\pmod 4$, for even subscripts $\neq n-2, n$. Hence, by considering residue classes modulo 4, we see that $v_{2i-1}\not\leftrightarrow v_{2j-1}$ for $i\neq j$ and $v_{2i}\not\leftrightarrow v_{2j}$ for $i\neq j$, $i,j\neq \frac{n}{2}-1,\frac{n}{2}$. Thus $v_{2i-1}\leftrightarrow v_{2j}$ if and only if $\ell_{2i-1}+\ell_{2j}\in\{2n-4,2n\}$, so that each of the vertices (possibly except v_{n-2} and v_n) must have degree 0, 1 or 2. Since these vertices lie on the path with labels given by Eq. (14), we need to show that $d(v_1)=d(v_n)=1$, $d(v_{n-2})=2$; the vertex with label $2n+3=\max L_2$ is isolated. We note that in order that v_i is a neighbour of v_1 (respectively, v_{n-2}, v_n), ℓ_i must belong to $\{1,4\}$ (respectively, $\{3,4,7\}$, $\{3\}$). The proof of the claim that the graph induced by $L_2\setminus\{2n+3\}$ is isomorphic to $\mathcal{P}_n$ is complete with the observation that $1,3,7\in L_2$ and $4\notin L_2$ when n is even.

Hence $\operatorname{spum}\mathcal{P}_n\le\max L_2-\min L_2=(2n+3)-1=2n+2$ when n is even. ■

Theorem 7.2. *For $n\ge 3$, $\texttt{spum}\,\mathcal{P}_n\ge 2n-3$.*

Proof. This follows directly from Theorem 2.1. ■

Theorem 7.3. *For $n\ge 7$,*

$$\texttt{integral spum}\,\mathcal{P}_n\le\begin{cases}\frac{5}{2}(n-3) & \text{if } n \text{ is odd;}\\ 2n-3 & \text{if } n \text{ is even.}\end{cases}$$

Proof. For odd $n\ge 7$, let G_1 be the graph induced by the set of labels $L_1=\mathcal{L}(G_1)=\left[-(n-3),-\frac{n-3}{2}\right]\cup\left\{\frac{n-3}{2},\frac{n-1}{2}\right\}\cup\left[n-2,\frac{3(n-3)}{2}\right]$. We claim that $G_1\cong\mathcal{P}_n$ by showing that the vertices with labels in the sequence

$$\tfrac{n-1}{2}, \tfrac{n-3}{2}, -(n-3), \tfrac{3n-9}{2}, -(n-4), \tfrac{3n-11}{2}, -(n-5), \tfrac{3n-13}{2}, \ldots, n-2, -\tfrac{n-3}{2} \qquad (15)$$

form a path. We note that the labels are given by $\ell_{2i} = \frac{3n-5}{2} - i$ for $i \in \{2, \ldots, \frac{n-1}{2}\}$ and $\ell_{2i+1} = -(n-2) + i$ for $i \in \{1, \ldots, \frac{n-1}{2}\}$, with $\ell_1 = \frac{n-1}{2}$ and $\ell_2 = \frac{n-3}{2}$. Observe that $v_1 \leftrightarrow v_2$, $v_2 \leftrightarrow v_3$ and $v_3 \leftrightarrow v_4$, and that $v_{2i} \leftrightarrow v_{2i+1}$ since $\ell_{2i} + \ell_{2i+1} = (\frac{3n-5}{2} - i) + (-n+2+i) = \frac{n-1}{2}$ for $i \in \{2, \ldots, \frac{n-1}{2}\}$.

We now prove that $v_i \not\leftrightarrow v_j$ when $|i-j| > 1$. Note that $\left\{\ell_{2i+1} : 1 \le i \le \frac{n-1}{2}\right\} = \left[-(n-3), -\frac{n-3}{2}\right] = I_1$ and that $\left\{\ell_{2i} : 2 \le i \le \frac{n-1}{2}\right\} = \left[n-2, \frac{3(n-3)}{2}\right] = I_2$. Let $S_1 + S_2 = \{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2, x_1 \neq x_2\}$. Then $I_1 + I_1 = [-(2n-7), -(n-2)] \cap L_1 = \emptyset$, $I_1 + I_2 = [1, n-3] \cap L_1 = \left\{\frac{n-3}{2}, \frac{n-1}{2}\right\}$, and $I_2 + I_2 = [2n-3, 3n-10] \cap L_1 = \emptyset$. Hence $v_i \leftrightarrow v_j$, $i, j \notin \{1, 2\}$ if and only if $\ell_i + \ell_j \in \left\{\frac{n-3}{2}, \frac{n-1}{2}\right\}$.

Suppose $\ell_{2i+1} = -(n-2) + i \in I_1$ and $v_{2i+1} \leftrightarrow v_j$. Then $\ell_j = \frac{3n-5}{2} - i$ or $\frac{3n-7}{2} - i$, $i \in \{1, \ldots, \frac{n-1}{2}\}$. Since both $\frac{3n-5}{2} - i$, $\frac{3n-7}{2} - i$ belong to I_2 for all $i \neq \frac{n-1}{2}$ while $\frac{3n-5}{2} - \frac{n-1}{2} \in I_2$, $\frac{3n-7}{2} - \frac{n-1}{2} \notin I_2$, each of the vertices v_{2i+1}, $i > 1$ has no neighbours aside from those on the path given by Eq. (15).

Suppose $\ell_{2i} = \frac{3n-5}{2} - i \in I_2$ and $v_{2i} \leftrightarrow v_j$. Then $\ell_j = -(n-2)+i$ or $-(n-3)+i$, $i \in \{2, \ldots, \frac{n-1}{2}\}$. Since both $-(n-2)+i$, $-(n-3)+i$ belong to I_2 for all i, each of the vertices v_{2i}, $i > 1$ has no neighbours aside from those on the path given by Eq. (15).

There remains to consider the vertices v_1 and v_2. Observe that $\left(I_1 + \frac{n-1}{2}\right) \cap L_1 = \left\{\frac{n-3}{2}\right\}$ and that $\left(I_1 + \frac{n-3}{2}\right) \cap L_1 = \left\{\frac{n-1}{2}, -(n-3)\right\}$. Thus there are no edges other than those on the path given by Eq. (15). This completes the proof of claim that $G_1 \cong \mathcal{P}_n$.

Hence integral spum $\mathcal{P}_n \le \max L_1 - \min L_1 = \frac{3}{2}(n-3) + (n-3) = \frac{5}{2}(n-3)$ when n is odd.

For even $n \ge 8$, let G_2 be the graph induced by the set of labels $L_2 = \mathcal{L}(G_2) = \{-1, 1, 3, \ldots, 2n-9\} \cup \{2n-8, 2n-5, 2n-4\}$. We claim that $G_2 \cong \mathcal{P}_n$ by showing that the vertices with labels in the sequence

$$2n-4, -1, 2n-8, 3, 2n-11, 7, 2n-15, 11, 2n-19, \ldots, 1, 2n-5 \qquad (16)$$

form a path. We note that the labels are given by $\ell_{2i} = 4(i-1) - 1 = 4i - 5$ for $i \in \{1, \ldots, \frac{n}{2}\}$ and $\ell_{2i-1} = (2n-11) - 4(i-3) = 2n + 1 - 4i$ for $i \in \{3, \ldots, \frac{n}{2}\}$, with $\ell(v_1) = 2n-4$ and $\ell(v_3) = 2n-8$. Observe that $v_1 \leftrightarrow v_2$, $v_2 \leftrightarrow v_3$, $v_3 \leftrightarrow v_4$ and $v_4 \leftrightarrow v_5$, and that $v_{2i-1} \leftrightarrow v_{2i}$ since $\ell_{2i-1} + \ell_{2i} = (2n+1-4i) + (4i-5) = 2n-4$ for $i \in \{3, \ldots, \frac{n}{2}\}$.

We now prove that $v_i \not\leftrightarrow v_j$ when $|i-j| > 1$. Observe that exactly two of the labels are even (in fact, multiples of 4), and these correspond to the vertices v_1 and v_3. Observe also that $\ell_{\mathrm{odd}} \equiv 1 \pmod 4$, for odd subscripts $\neq 1, 3$ and $\ell_{\mathrm{even}} \equiv 3 \pmod 4$. Suppose $v_i \leftrightarrow v_j$ with $i, j \notin \{1, 3\}$, $i \neq j$. Hence i and j must be of opposite parity, so that $(2n+1-4i) + (4j-5) \in \{2n-4, 2n-8\}$. But then $|i-j| \in \{0, 1\}$, and this is impossible. Since $v_1 \not\leftrightarrow v_4$, it remains to show that $v_1 \not\leftrightarrow v_i$ and $v_3 \not\leftrightarrow v_i$ for $i > 4$. Suppose, to the contrary, that $v_i \leftrightarrow v_1$ or $v_i \leftrightarrow v_3$. Then ℓ_i added to one of $2n-4$, $2n-8$ must belong to $\{\ell_j : j \equiv i \pmod 2\}$. Since $\left|\ell_j - \ell_i\right| \le 2n-12$ under the constraints, we arrive at a contradiction. This completes the proof of claim that $G_2 \cong \mathcal{P}_n$.

Hence integral spum $\mathcal{P}_n \le \max L_2 - \min L_2 = (2n-4) + 1 = 2n-3$ when n is even. ■

Theorem 7.4. *For $n \ge 3$, integral spum $\mathcal{P}_n \ge 2n-5$.*

Proof. This follows directly from Theorem 2.2. ■

We close this section with a table of values for spum $\mathcal{P}_n$ for $4 \le n \le 9$ (Table 3) and integral spum $\mathcal{P}_n$ for $3 \le n \le 13$ (Table 4), and conjectures on their exact values based on limited numerical evidence. The largest labels in Table 3 are also the labels of the isolated vertex in each case.

Table 3
Table of results for spum of paths.

n	spum labelling	spum $\mathcal{P}_n$
4	3, **1**, 2, 4; **6**	5
5	5, **1**, 4, 2, 6; **8**	7
6	9, **1**, 4, 5, 2, 7; **10**	9
7	12, **1**, 6, 7, 2, 4, 9; **13**	12
8	12, **1**, 11, 5, 7, 9, 3, 13; **16**	15
9	19, **1**, 11, 9, 3, 12, 7, 5, 15; **20**	19
10	19, **1**, 15, 5, 11, 9, 7, 16, 3, 20; **23**	22
11	23, **1**, 15, 9, 7, 16, 3, 13, 11, 5, 19; **24**	23
12	23, **1**, 19, 5, 15, 9, 11, 13, 7, 20, 3, 24; **27**	26
13	27, **1**, 19, 9, 11, 17, 3, 20, 7, 13, 15, 5, 23; **28**	27
14	27, **1**, 23, 5, 19, 9, 15, 13, 11, 17, 7, 24, 3, 28; **31**	30

Table 4
Table of results for integral spum of paths.

n	integral spum labelling	integral spum $\mathcal{P}_n$
3	1, **0**, **2**	2
4	1, 2, −**1**, **3**	4
5	2, 1, 3, −**2**, **4**	6
6	2, −**4**, 1, −3, **4**, −2	8
7	3, 2, −**4**, **6**, −3, 5, −2	10
8	**12**, −**1**, 8, 3, 5, 7, 1, 11	13
9	4, 3, −**6**, **9**, −5, 8, −4, 7, −3	15
10	**16**, −**1**, 12, 3, 9, 7, 5, 11, 1, 15	17
11	5, 4, −**8**, **12**, −7, 11, −6, 10, −5, 9, −4	20
12	**20**, −**1**, 16, 3, 13, 7, 9, 11, 5, 15, 1, 19	21
13	6, 5, −**10**, **15**, −9, 14, −8, 13, −7, 12, −6, 11, −5	25

Conjecture 7.1. *For* $n \geq 9$,

$$\operatorname{spum} \mathcal{P}_n = \begin{cases} 2n+1 & \text{if } n \text{ is odd;} \\ 2n+2 & \text{if } n \text{ is even.} \end{cases}$$

Conjecture 7.2. *For* $n \geq 7$,

$$\texttt{integral spum } \mathcal{P}_n = \begin{cases} \frac{5}{2}(n-3) & \text{if } n \text{ is odd;} \\ 2n-3 & \text{if } n \text{ is even.} \end{cases}$$

CRediT authorship contribution statement

Sahil Singla: Conceptualization, Methodology, Formal analysis, Investigation, Writing - original draft, Writing - review & editing. **Apurv Tiwari:** Conceptualization, Methodology, Formal analysis, Investigation, Writing - original draft, Writing - review & editing. **Amitabha Tripathi:** Conceptualization, Investigation, Writing - original draft, Writing - review & editing, Supervision, Project administration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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